## 8.5 Numerical Evaluation of Probabilities

1

Density of event individual became disabled at time $t$ is

$$ {}_tp^{00}_{27}\mu^{01}_{27+t}\,{}_{16-t}p^{11}_{27+t} $$

so probability is

$$ \int_0^{16} 0.003e^{-0.004t}e^{-0.002(16-t)}dt = 0.003e^{-0.032}\int_0^{16} e^{-0.002t}dt = \frac{0.003}{0.002}e^{-0.032}(1-e^{-0.032}) = 0.04575237 $$

2

Density of event individual became disabled at time $t$ is

$$ {}_tp^{00}_{32}\mu^{01}_{32+t}\,{}_{12-t}p^{11}_{32+t} $$

We have

$$ \begin{aligned} {}_tp^{00}_x &= e^{-\int_0^t 0.004+0.00003(x+s)ds} = e^{-(0.004+0.00003x)t+0.000015t^2} \\ {}_tp^{11}_x &= e^{-\int_0^t 0.002+0.00002(x+s)ds} = e^{-(0.002+0.00002x)t+0.00001t^2} \end{aligned} $$

so probability is

$$ \begin{aligned} &\int_0^{12}(0.001+0.00001(32+t))e^{-(0.00496t+0.000015t^2)}e^{-(0.00264(12-t)+0.00001(12-t)^2)}dt \\ &= \int_0^{12}(0.001+0.00001(32+t))e^{-(0.00496t+0.000015t^2)}e^{-((0.00264+0.00002t)(12-t)+0.00001(12-t)^2)}dt \\ &= \int_0^{12}(0.001+0.00001(32+t))e^{-(0.03312+0.0028t+0.000005t^2)}dt \\ &= \int_0^{12}(0.001+0.00001(32+t))e^{-\frac{((t+28)^2)-121.6}{20000}}dt \\ &= \frac{e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}(132+t)e^{-\frac{((t+28)^2)}{20000}}dt \\ &= \frac{e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}(104+(t+28))e^{-\frac{((t+28)^2)}{20000}}dt \\ &= \frac{104e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}e^{-\frac{((t+28)^2)}{20000}}dt + \frac{e^{-\frac{121.6}{20000}}}{10000}\left[-10000e^{-\frac{((t+28)^2)}{20000}}\right]_0^{12} \\ &= \frac{104}{100}e^{-\frac{121.6}{20000}}\sqrt{2\pi}\left(\Phi\left(\frac{28}{100}\right)-\Phi\left(\frac{16}{100}\right)\right)+e^{-\frac{121.6}{20000}}\left(e^{-\frac{28^2}{20000}}-e^{-\frac{40^2}{20000}}\right) \\ &= 0.1250398 \end{aligned} $$

3

We have

$$
\begin{aligned}
{}_tp_x^{\overline{00}} &= e^{-0.0004t} \\
{}_tp_x^{\overline{11}} &= e^{-0.00023t}
\end{aligned}
$$

Probability this happens from 1 transition:

$$
\begin{aligned}
&\int_0^{16} {}_tp_{27}^{\overline{00}} \mu_{27+t}^{01} {}_{(16-t)}p_{27+t}^{\overline{11}}\, dt \\
&= \int_0^{16} 0.0003 e^{-0.0004t} e^{-0.00023(16-t)}\, dt \\
&= 0.0003 e^{-0.00368} \int_0^{16} e^{-0.00017t}\, dt \\
&= \frac{0.0003}{0.00017} e^{-0.00368} \left(1 - e^{-0.00272}\right)
\end{aligned}
$$

Probability this happens from 3 transitions:

$$
\begin{aligned}
&\int_0^{16}\int_s^{16}\int_t^{16} {}_sp_{27}^{\overline{00}} \mu_{27+s}^{01} {}_{(t-s)}p_{27+s}^{\overline{11}} \mu_{27+t}^{10} {}_{u-t}p_{27+t}^{\overline{00}} \mu_{27+u}^{01} {}_{(16-u)}p_{27+u}^{\overline{11}}\, du\, dt\, ds \\
&= \int_0^{16}\int_s^{16}\int_t^{16} 0.0003^2 \times 0.00003 e^{-0.0004(s+u-t)} e^{-0.00023(16-u+t-s)}\, du\, dt\, ds \\
&= 0.0003^2 \times 0.00003 e^{-0.00368} \int_0^{16}\int_s^{16}\int_t^{16} e^{-0.00017(s+u-t)}\, du\, dt\, ds
\end{aligned}
$$

Making the substitution $w = s + u - t$, this becomes (noting that $|J| = 1$):

$$
\begin{aligned}
&0.0003^2 \times 0.00003 e^{-0.00368} \int_0^{16}\int_0^{w}\int_s^{16+s-w} e^{-0.00017w}\, dt\, ds\, dw \\
&0.0003^2 \times 0.00003 e^{-0.00368} \int_0^{16} e^{-0.00017w} \int_0^{w}\int_s^{16+s-w} 1\, dt\, ds\, dw \\
&0.0003^2 \times 0.00003 e^{-0.00368} \int_0^{16} w(16-w) e^{-0.00017w}\, dw
\end{aligned}
$$

In general, the probability that this happens from $2n+1$ transitions is

$$
0.0003 \left(0.00003 \times 0.0003\right)^n e^{-0.00368} \int_0^{16} \frac{w^n (16-w)^n}{(n!)^2} e^{-0.00017w}\, dw
$$

The total probability of this is therefore

$$\sum_{n=0}^{\infty} 0.0003\,(0.00003\times 0.0003)^n\, e^{-0.00368}\int_0^{16}\frac{w^n(16-w)^n}{(n!)^2}e^{-0.00017w}\,dw$$
$$= 0.004775872$$

## 8.6 Premiums

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The rate of exit of state 0 is $0.0004+0.000003(t+42)=0.000526+0.000003t$, so ${}_tp_{42}^{(00)}=e^{-\int_0^t 0.000526+0.000003t\,dt}=e^{-0.000526-0.0000015t^2}$.

Premiums are payable while healthy. If the rate of premium is $P$, then the expected present value of the premium paid is

$$\begin{aligned}
&P\int_0^5 e^{-0.03t}e^{-0.000526t-0.0000015t^2}\,dt\\
&=P\int_0^5 e^{-0.030526t-0.0000015t^2}\,dt\\
&=P\int_0^5 e^{-0.0000015(t+10175.333333)^2+0.0000015\times 10175.333333^2}\,dt\\
&=Pe^{0.0000015\times 10175.333333^2}\sqrt{\frac{\pi}{0.0000015}}\left(\Phi\left(10180.333333\sqrt{0.000003}\right)-\Phi\left(10175.333333\sqrt{0.000003}\right)\right)\\
&=4.637064P
\end{aligned}$$

On the other hand, the expected death benefits for lives that are not critically ill first are:

$$\begin{aligned}
&100000\int_0^5 e^{-0.03t}e^{-0.000526t-0.0000015t^2}(0.0001+0.000001(t+42))\,dt\\
&=100000\int_0^5 e^{-0.030526t-0.0000015t^2}(0.000142+0.000001t)\,dt\\
&=100000\int_0^5 e^{-0.030526t-0.0000015t^2}(0.000001(t+\frac{0.030526}{0.000003})+0.000142-0.01017533333333)\,dt\\
&=0.1e^{\frac{0.030526^2}{0.000006}}\int_0^5\left(t+\frac{0.030526}{0.000003}\right)e^{-0.0000015\left(t+\frac{0.030526}{0.000003}\right)^2}\,dt-1003.333333333333\int_0^5 e^{-0.030526t-0.0000015t^2}\,dt\\
&=0.1e^{\frac{0.030526^2}{0.000006}}\left[-\frac{e^{-0.0000015\left(t+\frac{0.030526}{0.000003}\right)^2}}{0.000003}\right]_0^5-1003.333333333333\times 4.637064\\
&=66.97587
\end{aligned}$$

If a life becomes critically ill at age $x$, the probability that it survives for $t$ years is $e^{-\int_0^t 0.02\,dt} = e^{-0.02t}$. The expected value of the benefits to such a life is therefore given by

$$90000\int_0^{47-x} e^{-0.03(x+t-42)}e^{-0.02t}\,dt = 90000e^{-0.03(x-42)}\int_0^{47-x} e^{-0.05t}\,dt = 1800000e^{-0.03(x-42)}(1-e^{-0.05(47-x)})$$

The expected value of death benefits to such an individual is

$$100000\int_0^{47-x} 0.02e^{-0.03(x+t-42)}e^{-0.02t}\,dt = 2000e^{-0.03(x-42)}\int_0^{47-x} e^{-0.05t}\,dt = 4000e^{-0.03(x-42)}(1-e^{-0.05(47-x)})$$

So the total expected benefits paid to individuals who become disabled are

$$\begin{aligned}
&1840000\int_0^5 e^{-0.000526s-0.0000015s^2}(0.0003+0.000002(s+42))e^{-0.03(5-s)}(1-e^{-0.05(5-s)})\,ds\\
&= 1.84\int_0^5 (384+2s)e^{-0.15+0.029474s-0.0000015s^2}(1-e^{-0.05(5-s)})\,ds\\
&= 1.84\int_0^5 (384+2s)(e^{-0.15+0.029474s-0.0000015s^2} - e^{-0.4+0.079474s-0.0000015s^2})\,ds
\end{aligned}$$

$$\begin{aligned}
0.15-0.029474s+0.0000015s^2 &= 0.0000015(s^2-19649.333333s+100000)\\
&= 0.0000015((s-9824.666667)^2-96424075)\\
0.4-0.079474s+0.0000015s^2 &= 0.0000015(s^2-52982.666667s+266666.666667)\\
&= 0.0000015((s-26491.333333)^2-701524075)
\end{aligned}$$

So the expected benefits are

$$\begin{aligned}
&1.84\int_0^5 (384+2s)(e^{144.6361}e^{-\frac{0.000003(s-9824.666667)^2}{2}} - e^{1052.286}e^{-\frac{0.000003(s-26491.333333)^2}{2}})\,ds\\
&= 1.84e^{144.6361}\int_0^5 (20033.3333333+2(s-9824.67))e^{-\frac{0.000003(s-9824.67)^2}{2}}\,ds - 1.84e^{1052.286}\int_0^5 (53366.67+2(s-26491.33))e^{-\frac{0.000003(s}{}}\\
&= 1.84e^{144.6361}\left(20033.33\times\sqrt{\frac{2\pi}{0.000003}}\left(\Phi(-9819.67\times\sqrt{0.000003})-\Phi(-9824.67\times\sqrt{0.000003})\right)-2\left[\frac{e^{-\frac{0.000003(s-9824.67)^2}{2}}}{0.000003}\right]_0^5\right)\\
&\quad -1.84e^{1052.286}\left(53366.67\times\sqrt{\frac{2\pi}{0.000003}}\left(\Phi(-26491.33\times\sqrt{0.000003})-\Phi(-26496.33\times\sqrt{0.000003})\right)-2\left[\frac{e^{-\frac{0.000003(s-26491.33)^2}{2}}}{0.000003}\right]_0^5\right.\\
&= 371.8703
\end{aligned}$$

The total expected benefit is therefore

$$371.8703 + 66.97587 = 438.8461$$

The premium is therefore

$$\frac{438.8461}{4.637572} = \$94.64$$

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$$0.000001 \begin{pmatrix} -\left(1843x + 38x^2 + \frac{x^3}{3}\right) & 374x + x^2 & 1469x + 37x^2 + \frac{x^3}{3} \\ 67x + 0.5x^2 & -(341x + 1.5x^2) & 274x + x^2 \\ 0 & 0 & 0 \end{pmatrix}$$

We calculate the probability that the life is in each state at the end of each year:

| $t$ | ${}_tp_{37}^{00}$ | ${}_tp_{37}^{01}$ | ${}_tp_{37}^{02}$ |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0.99812 | 0.000375 | 0.001505 |
| 2 | 0.99617 | 0.000750 | 0.003083 |
| 3 | 0.99414 | 0.001127 | 0.004736 |
| 4 | 0.99203 | 0.001505 | 0.006464 |
| 5 | 0.98985 | 0.001884 | 0.008271 |
| 6 | 0.98758 | 0.002263 | 0.010156 |
| 7 | 0.98523 | 0.002644 | 0.012123 |
| 8 | 0.98280 | 0.003025 | 0.014171 |
| 9 | 0.98029 | 0.003407 | 0.016303 |
| 10 | 0.97769 | 0.003790 | 0.018519 |

EPV of death benefits is

$$200000(0.00150(1.06)^{-1}+0.00157(1.06)^{-2}+0.00165(1.06)^{-3}+0.00172(1.06)^{-4}+0.00180(1.06)^{-5}+0.00188(1.06)^{-6}+0.00197(1.06)^{-7}+$$

EPV of disability benefits is

$$80000(0.000375(1.06)^{-1}+0.000750(1.06)^{-2}+0.001127(1.06)^{-3}+0.001505(1.06)^{-4}+0.001884(1.06)^{-5}+0.002263(1.06)^{-6}+0.002644(1.0$$

EPV of all benefits is $2670.49 + 1116.21 = 3786.70$

EPV of unit premiums is $1 + 0.99812(1.06)^{-1} + 0.99617(1.06)^{-2} + 0.99414(1.06)^{-3} + 0.99203(1.06)^{-4} + 0.98985(1.06)^{-5} + 0.98758(1.06)^{-6} + 0.98523(1.06)^{-7} + 0.98280(1.06)^{-8} + 0.98029(1.06)^{-9} = 7.736653311$

So annual premium is

$$\frac{3786.70}{7.736653311} = \$489.45$$

## 8.7 Policy Values and Thiele's Differential Equation

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Thiele's differential equation gives

$$\begin{aligned}
\frac{d}{dt}{}_tv^{(1)} &= \delta_t v^{(1)} + P^{(1)} - B^{(1)} - \sum_{j\neq 1}\mu_{x+t}^{(1j)}(S^{(1j)} +_t v^{(j)} -_t v^{(1)}) \\
&= 0.03_t v^{(1)} - 90000 - \mu_{x+t}^{(12)}(100000 - {}_tv^{(1)}) \\
&= 0.03_t v^{(1)} - 90000 - 0.02(100000 - {}_tv^{(1)}) \\
&= 0.05_t v^{(1)} - 92000 \\
{}_tv^{(1)} &= 1840000(1 - e^{0.05(t-5)})
\end{aligned}$$

$$\begin{aligned}
\frac{d}{dt}{}_tv^{(0)} &= \delta_t v^{(0)} + P^{(0)} - B^{(0)} - \sum_{j\neq 0}\mu_{x+t}^{(0j)}(S^{(0j)} +_t v^{(j)} -_t v^{(0)}) \\
&= 0.03_t v^{(0)} + 71.83 - 0 - \mu_{x+t}^{(01)}({}_tv^{(1)} -_t v^{(0)}) - \mu_{x+t}^{(02)}(100000 - {}_tv^{(0)}) \\
&= 0.03_t v^{(0)} + 71.83 - (0.0003 + 0.000002(42+t))({}_tv^{(1)} -_t v^{(0)}) - (0.0001 + 0.000001(42+t))(100000 - {}_tv^{(0)}) \\
&= 0.03_t v^{(0)} + 71.83 - (0.000384 + 0.000002t)(1840000(1 - e^{0.05(t-5)}) -_t v^{(0)}) - (0.000142 + 0.000001t)(100000 - {}_tv^{(0)}) \\
&= 71.83 - 706.56 - 14.2 - 3.68t(1 - e^{0.05(t-5)}) - 0.1t + (0.03 + 0.0000384 + 0.000142)_t v^{(0)} + 0.000003t_t v^{(0)} + 70.656e^{0.05(t-5} \\
&= -13.026 - 3.78t + 3.68te^{0.05(t-5)} + 0.0300526_t v^{(0)} + 0.000003t_t v^{(0)} + 706.56e^{0.05(t-5)}
\end{aligned}$$

## 8.9 Multiple Decrement Tables

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We have ${}_tp_x^{(00)} = e^{-0.0023t - 0.000005((x+t)^2 - x^2)}$, so we have

$$\begin{aligned}
\overline{a}_{36:\overline{10}|}^{(00)} &= \int_0^{10} e^{-0.04t} e^{-0.0023t - 0.000005((36+t)^2 - 36^2)}\,dt \\
&= \int_0^{10} e^{-0.04266t - 0.000005t^2}\,dt \\
&= e^{\frac{4266}{200000}} \int_0^{10} e^{-\frac{(t+4266)^2}{200000}}\,dt \\
&= 100\sqrt{20\pi}e^{\frac{4266^2}{20000}}\left(\Phi\left(\frac{42.76}{\sqrt{10}}\right) - \Phi\left(\frac{42.66}{\sqrt{10}}\right)\right) = 8.139325
\end{aligned}$$

$$
\begin{aligned}
\overline{A}^{(01)}_{36:\overline{10}|} &= \int_0^{10} (0.00102 + 0.00002t)e^{-0.04t}e^{-0.0023t-0.000005((36+t)^2-36^2)}\,dt \\
&= e^{\frac{4266^2}{200000}} \int_0^{10} 0.00002(t+51)e^{-\frac{(t+4266)^2}{200000}}\,dt \\
&= 0.00002e^{\frac{4266^2}{200000}} \left( \int_0^{10} (t+4266)e^{-\frac{(t+4266)^2}{200000}}\,dt - 4215\int_0^{10} e^{-\frac{(t+4266)^2}{200000}}\,dt \right) \\
&= 0.00002e^{\frac{4266^2}{200000}} \left( \left[ -100000e^{-\frac{(t+4266)^2}{200000}} \right]_0^{10} - 421500\sqrt{20\pi}\left( \Phi\left(\frac{42.76}{\sqrt{10}}\right) - \Phi\left(\frac{42.66}{\sqrt{10}}\right) \right) \right) \\
&= 0.009058283
\end{aligned}
$$

So the annual rate of premium is

$$
\frac{300000 \times 0.01093749}{8.007432} = \$333.87
$$

(b) For policies which do not lapse, we only need to consider the mortality rate. This gives

$$
\begin{aligned}
\overline{a}^{(00)}_{36+t:\overline{10-t}|} &= \int_0^{10-t} e^{-0.04s}e^{-0.0003s-0.000005((36+t+s)^2-(36+t)^2)}\,ds \\
&= \int_0^{10-t} e^{-(0.04102+0.00001t)s-0.000005s^2}\,ds \\
&= e^{\frac{4102^2}{200000}} \int_0^{10-t} e^{-\frac{(s+t+4102)^2}{200000}}\,ds \\
&= 100\sqrt{20\pi}e^{\frac{(t+4102)^2}{200000}} \left( \Phi\left(\frac{41.12}{\sqrt{10}}\right) - \Phi\left(\frac{t+4102}{100\sqrt{10}}\right) \right)
\end{aligned}
$$

and

$$
\overline{A}^{(00)}_{36+t:\overline{10-t}|} = 1 - 0.04\overline{a}^{(00)}_{36+t:\overline{10-t}|}
$$

So the policy value is

$$
\begin{aligned}
{}_tV &= 300000(1 - 0.04\overline{a}^{(00)}_{36+t:\overline{10-t}|} - {}_{10-t}\,p_{36+t}e^{-0.04(10-t)}) - P\overline{a}^{(00)}_{36+t:\overline{10-t}|} \\
&= 300000 - (12000+P)\overline{a}^{(00)}_{36+t:\overline{10-t}|} - 300000\,{}_{10-t}p_{36+t}e^{-0.04(10-t)} \\
&= 300000 - (12000+P)100\sqrt{20\pi}e^{\frac{(t+4102)^2}{200000}} \left( \Phi\left(\frac{41.12}{\sqrt{10}}\right) - \Phi\left(\frac{t+4102}{100\sqrt{10}}\right) \right) - 300000\,{}_{10-t}p_{36+t}e^{-0.04(10-t)} \\
&= a(t) + b(t)P
\end{aligned}
$$

for some functions $a(t)$ and $b(t)$.

From (a), the expected death benefit of the policy is $300000 \times 0.01093749 = \$3281.25$.

The expected surrender benefit for the policy is

$$\begin{aligned}
\frac{1}{2}\int_0^{10} e^{-0.04t}{}_tp_{36}\mu^{(01)}_{36+t}{}_tV\,dt &= \frac{1}{2}\int_0^{10} e^{-0.04t}e^{-(0.0023t+0.000005(t^2+72t))}(0.002-0.00001(36+t))_tV\,dt\\
&= \frac{1}{2}\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))_tV\,dt\\
&= \frac{1}{2}\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))(a(t)+b(t)P)\,dt\\
&= \frac{1}{2}\left(\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))a(t)\,dt + \int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))b(t)P\right.\\
&= \frac{1}{2}(46.51242-0.08175562P)
\end{aligned}$$

Using numerical integration. We note that this requires the policy value to always be positive.

We therefore have

$$\begin{aligned}
8.007432P &= 3281.247+23.25621-0.04087781P\\
7.966554P &= 3304.50321\\
P &= \$414.80
\end{aligned}$$

8

We have

$$ {}_tp_x^{(00)} = e^{-\int_0^t 0.0023+0.00001(x+s)\,ds} = e^{-((0.0023+0.00001x)t+0.000005t^2)}$$

In particular,

$$ {}_tp_{29}^{(00)} = e^{-(0.00259t+0.000005t^2)}$$

We therefore calculate

$$\begin{aligned}
\overline{a}^{(00)}_{29:\overline{10}|} &= \int_0^{10} e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\,dt\\
&= \int_0^{10} e^{-(0.05259t+0.000005t^2)}\,dt\\
&= e^{\frac{5259^2}{200000}}\int_0^{10} e^{-\frac{(t+5259)^2}{200000}}\,dt\\
&= e^{\frac{5259^2}{200000}}100\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right)\\
&= 7.775573
\end{aligned}$$

On the other hand we have

$$
\begin{aligned}
\overline{A}^{(01)}_{29:\overline{10}|} &= \int_0^{10} 0.0003e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\,dt \\
&= 0.0003\int_0^{10} e^{-(0.05259t+0.000005t^2)}\,dt \\
&= 0.0003e^{\frac{5259^2}{200000}}\int_0^{10} e^{-\frac{(t+5259)^2}{200000}}\,dt \\
&= 0.0003e^{\frac{5259^2}{200000}}100\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right) \\
&= 0.002332672
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{A}^{(02)}_{29:\overline{10}|} &= \int_0^{10} 0.00002(29+t)e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\,dt \\
&= \int_0^{10} 0.00002(29+t)e^{-(0.05259t+0.000005t^2)}\,dt \\
&= 0.00002e^{\frac{5259^2}{200000}}\int_0^{10}(t+5259-5230)e^{-\frac{(t+5259)^2}{200000}}\,dt \\
&= 0.00002e^{\frac{5259^2}{200000}}\left(100000\left(e^{-\frac{(5259)^2}{200000}}-e^{-\frac{(5269)^2}{200000}}\right)-523000\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right)\right) \\
&= 0.005219487
\end{aligned}
$$

The EPV of benefits is therefore

$$100000\times 0.005219487+200000\times 0.002332672 = 988.48$$

and the premium is

$$\frac{988.48}{7.775573} = \$127.13$$

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$$\frac{56.94+56.61}{9878.44} = 0.01149473$$

10

We calculate

$$
\begin{aligned}
A^{(02)}{}^{\,1}_{40:\overline{5}|} &= \frac{0.30(1.03)^{-1}+0.29(1.03)^{-2}+0.28(1.03)^{-3}+0.27(1.03)^{-4}+0.27(1.03)^{-5}}{10000} = 0.000129365 \\
A^{(03)}{}^{\,1}_{40:\overline{5}|} &= \frac{1.62(1.03)^{-1}+1.70(1.03)^{-2}+1.78(1.03)^{-3}+1.89(1.03)^{-4}+1.98(1.03)^{-5}}{10000} = 0.0008191387 \\
\ddot{a}_{40:\overline{5}|} &= 1+0.993908(1.03)^{-1}+0.987844(1.03)^{-2}+0.981806(1.03)^{-3}+0.975795(1.03)^{-4} = 4.66157
\end{aligned}
$$

So the net premium is $\frac{200000\times 0.000129365+100000\times 0.0008191387}{4.66157}=\$23.12$

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(a) Using UDD in the multiple decrement table, the probability that the policy is still in force at age 46 years 3 months is $\frac{\frac{1}{4}9579.02+\frac{3}{4}0.963844}{\frac{1}{3}10000+\frac{2}{3}9939.08}=0.9662829$.

The probability that the life then dies in an accident before age 47 is $\frac{3}{4}\times\frac{0.25}{\frac{1}{4}\times 9579.02+\frac{3}{4}\times 9638.44}=0.00001948338$.

The probability that the life dies in an accident between ages 46 and 3 months and 47 is therefore $0.9662829\times 0.00001948338=0.00001882646$.

The probability that the life survives to age 47 is $\frac{9579.02}{\frac{1}{3}10000+\frac{2}{3}9939.08}=0.9618082$.

The probability that a life aged 47 dies in an accident before age 47 and 5 months is $\frac{5}{12}\times\frac{0.24}{9579.02}$. The probability that the life dies in an accident between ages 47 and 47 and 5 months is therefore $\frac{5}{12}\times\frac{0.24}{9579.02}\times\frac{9579.02}{\frac{1}{3}10000+\frac{2}{3}9939.08}=0.00001004078$.

The total probability is therefore

$0.00001882646+0.00001004078=0.00002886724$.

(b)

Using constant transition intensities, the rate of decrement for a life aged 42–43 is $-\log\left(\frac{9939.08}{10000}\right)=0.006110632$. The probability that a policy starting at age 42 is still in force at age 42 and 4 months is $e^{-\frac{0.006110632}{3}}=0.9979652$. The intensity of decrement aged 46–47 is $-\log\left(\frac{9579.02}{9638.44}\right)=0.00618398$. This gives that $l^{(0)}_{46.25}=9638.44e^{-0.25\times 0.00618398}=9623.55$. The probability that the policy is still in force at age 46 and 3 months is therefore $\frac{9623.55}{9979.65}=0.9643174$. The intensity of accidental deaths between ages 46–47 is

$$\frac{0.25}{9638.44\int_0^1 e^{-0.00618398x}\,dx}$$

and the probability of accidental death between ages 46 and 3 months and 47 is therefore

$$\frac{0.25}{9638.44\int_0^1 e^{-0.00618398x}\,dx}\int_0^{0.75} e^{-0.00618398x}\,dx=\frac{0.25}{9638.44}\frac{1-e^{-0.00618398\times 0.75}}{1-e^{-0.00618398}}=0.00001946839$$

The probability of surviving to age 47 is $\frac{9579.02}{9979.65}=0.9598551$.

The intensity of decrements between ages 47–48 is

$-\log\left(\frac{9519.81}{9579.02}\right)=0.0062004$.

The intensity of accidental deaths between ages 47–48 is

$$\frac{0.24}{9579.02\int_0^1 e^{-0.0062004x}\,dx}$$

and the probability of accidental death between ages 47 and 47 and 5 months is therefore

$$\frac{0.24}{9579.02\int_0^1 e^{-0.00620040.00618398x}\,dx}\int_0^{\frac{5}{12}} e^{-0.0062004x}\,dx=\frac{0.24}{9579.02}\frac{1-e^{-0.0062004\times\frac{5}{12}}}{1-e^{-0.0062004}}=0.00001045836$$

The total probability of dying in an accident between ages 46 years 3 months and 47 years 5 months is therefore

$$0.00001946839\times 0.9643174+0.00001045836\times 0.9598551=0.00002881222$$

## 8.10 Constructing a Multiple Decrement Table

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(a) Under UDD, suppose there are $x$ policies at the start of the year, and during the year, $y$ surrender and $z$ die. Assuming UDD, the number of policies still in force at time $t$ is $x-(y+z)t$, so the rate of dying is $\frac{z}{x-(y+z)t}$. Assume that the deaths in the new table are uniformly distributed over the year. (This is inconsistent with our uniform distribution assumption for policies with surrender, but what the hell.) If the updated mortality table has $w$ deaths from $v$ lives, then the rate of death at time $t$ is $\frac{v}{w-vt}$. With this update, let $x_t$ be the number of policies still in force at time $t$. We have

$$\frac{dx_t}{dt} = -\left(\frac{y}{x-(y+z)t} + \frac{v}{w-vt}\right)x_t$$

This gives

$$\begin{aligned} x_1 &= xe^{-\int_0^1 \frac{y}{x-(y+z)t}+\frac{v}{w-vt}\,dt} = e^{[y\log(x-(y+z)t)+v\log(w-vt)]_0^1} = e^{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)+\log\left(\frac{w-v}{w}\right)} \\ &= \left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right) \end{aligned}$$

In the new table, we get

$$\begin{aligned} \left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}} &= \left(\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)\right)^{\frac{y'}{y'+z'}} \\ \frac{y'}{y'+z'} &= \frac{y}{y+z}\left(\frac{\log\left(\frac{x-(y+z)}{x}\right)}{\log\left(\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)\right)}\right) \\ &= \frac{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)}{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)+\log\left(\frac{w-v}{w}\right)} \end{aligned}$$

Using this, we can calculate the probabilities on the following slide.

(b) Suppose there are $x$ policies at the start of the year, and during the year, $y$ surrender and $z$ die. Under constant transition probabilities, we have the number of policies still in force at time $t$ is $xe^{\log\left(\frac{x-y-z}{x}\right)t} = x\left(\frac{x-y-z}{x}\right)^t$. If the constant rate of surrender is $\mu^{(01)}$, then

$$y = \int_0^1 \mu^{(01)}xe^{\log\left(\frac{x-y-z}{x}\right)t}\,dt = \mu^{(01)}x\frac{1-\frac{x-y-z}{x}}{\log(x)-\log(x-y-z)} = \mu^{(01)}\frac{y+z}{\log(x)-\log(x-y-z)}$$

This gives $\mu^{(01)} = \frac{y(\log(x)-\log(x-y-z))}{(y+z)}$, so without deaths, the probability of surrender is $e^{-(\log(x)-\log(x-y-z))\frac{y}{(y+z)}} = \left(\frac{x-y-z}{x}\right)^{\frac{y}{y+z}}$. The table is therefore the same as in part (a).

(c) If each independent decrement satisfies UDD, then suppose the probabilities for surrender and death as only decrements are $p$ and $q$ respectively, then the rate of surrender is $\frac{p}{1-pt}$ and the rate of death is $\frac{q}{1-qt}$.

Now in a model with two decrements, the total rate of decrement is $\frac{p}{1-pt}+\frac{q}{1-qt}$, so the total probability of no decrement in the year is

$$e^{-\int_0^1 \frac{p}{1-pt}+\frac{q}{1-qt}\,dt} = e^{[\log(1-pt)+\log(1-qt)]_0^1} = (1-p)(1-q)$$

Similarly, the probability of no decrement before time $t$ is $(1-pt)(1-qt)$. The probability of surrender is

$$\int_0^1 (1-pt)(1-qt)\frac{p}{1-pt}\,dt = \int_0^1 p(1-qt)\,dt = p\left(1-\frac{q}{2}\right)$$

Similarly, the probability of death is $q\left(1-\frac{p}{2}\right)$.

If we are given the probabilities of surrender and death in the multiple decrement model are $a$ and $b$ respectively, then we have to solve

$$\begin{aligned}
p\left(1-\frac{q}{2}\right) &= a\\
q\left(1-\frac{p}{2}\right) &= b\\
p-q &= a-b\\
p-\frac{p^2}{2} &= a-\frac{p}{2}(a-b)\\
p^2+(b-a-2)p+2a &= 0\\
p &= \frac{a+2-b\pm\sqrt{(a+2-b)^2-8a}}{2}\\
&= \frac{a+2-b\pm\sqrt{a^2+b^2+4-2ab-4a-4b)}}{2}\\
q &= \frac{b+2-a\pm\sqrt{a^2+b^2+4-2ab-4a-4b)}}{2}
\end{aligned}$$

| $x$ | $l_x$ | $d_x$ |
|---|---|---|
| 40 | 10000.00 | 59.01 |
| 41 | 9940.99 | 59.02 |
| 42 | 9881.98 | 59.03 |
| 43 | 9822.95 | 59.04 |
| 44 | 9763.91 | 59.06 |
| 45 | 9704.85 | 59.07 |
| 46 | 9645.78 | 59.08 |
| 47 | 9586.69 | 59.11 |

## 9.2 Joint Life and Last Survivor Benefits

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| Advantages | Disadvantages |
|---|---|
| Annuity value does not depend on time of death | Value of benefit varies with time of death |

## 9.4 Independent Future Lifetimes

15

$$\ddot{a}_{63,62:\overline{10}|} = 1+0.999830\frac{9996.47}{9998.30}(1.07)^{-1}+0.999647\frac{999449}{9998.30}(1.07)^{-2}+0.999449\frac{9992.35}{9998.30}(1.07)^{-3}+0.999235\frac{9990.03}{9998.30}(1.07)^{-4}+0.999003$$

$$\begin{aligned}
\ddot{A}_{\overline{63,62}:\overline{10}|} &= 0.0000170 \times \frac{1.83}{9998.30}(1.07)^{-1} + \left(0.0000353 \times \frac{1.83+1.98}{9998.30} - 0.0000170 \times \frac{1.83}{9998.30}\right)(1.07)^{-2} \\
&+ \left(0.0000551 \times \frac{1.83+1.98+2.14}{9998.30} - 0.0000353 \times \frac{1.83+1.98}{9998.30}\right)(1.07)^{-3} \\
&+ \left(0.0000765 \times \frac{1.83+1.98+2.14+2.31}{9998.30} - 0.0000551 \times \frac{1.83+1.98+2.14}{9998.30}\right)(1.07)^{-4} \\
&+ \left(0.0000996 \times \frac{1.83+1.98+2.14+2.31+2.50}{9998.30} - 0.0000765 \times \frac{1.83+1.98+2.14+2.31}{9998.30}\right)(1.07)^{-5} \\
&+ \left(0.0001246 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70}{9998.30} - 0.0000996 \times \frac{1.83+1.98+2.14+2.31+2.50}{9998.30}\right)(1.07)^{-6} \\
&+ \left(0.0001516 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92}{9998.30} - 0.0001246 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70}{9998.30}\right)(1. \\
&+ \left(0.0001808 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16}{9998.30} - 0.0001516 \times \frac{1.83+1.98+2.14+2.31+2.50+2.}{9998.30} \right. \\
&+ \left(0.0002124 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16+3.41}{9998.30} - 0.0001808 \times \frac{1.83+1.98+2.14+2.31+2.}{9998.30} \right. \\
&+ \left(0.0002465 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16+3.41+3.69}{9998.30} - 0.0002124 \times \frac{1.83+1.98+2.14+2.}{} \right. \\
&= 0.0000003531257
\end{aligned}$$

So the premium is $\frac{2000000 \times 0.0000003531257}{7.502352} = \$0.09$.

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If the Husband is dead and the wife is alive at the end of the policy with probability $0.997567 \times 0.001251$, then the wife receives a reversionary annuity with value $30000 \times 13.89755 = 416926.5$. The expected present value of payments after the end of the policy is therefore $30000 \times 13.89755 \times 0.997567 \times 0.001251(1.05)^{-7} = 369.7718$. Payments during the term of the policy are received if and only if the husband is dead and the wife is alive. The expected present value of payments received during the term of the policy is therefore

$$30000\left(0.000135 \times 0.999736(1.05)^{-1} + 0.000283 \times 0.999448(1.05)^{-2} + 0.000444 \times 0.999134(1.05)^{-3} + 0.000620 \times 0.998791(1.05)^{-4} + \right.$$

So the total expected benefit is $369.7718 + 80.22267 = 449.9945$.

For the premiums, we have

$$\ddot{a}_{53,64:\overline{7}|} = 1+0.999865\times0.999736(1.05)^{-1}+0.999717\times0.999448(1.05)^{-2}+0.999555\times0.999134(1.05)^{-3}+0.999380\times0.998791(1.05)^{-4}+0$$

So the net annual premium is $\frac{80.22267}{6.067799} = \$13.22$.

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The probabilities of each being dead after $n$ years are:

| Years | P(Husband Dead) | P(Wife Dead) | P(Survivor) | EPV benefit |
|---|---|---|---|---|
| 0 | 0.000000 | 0.000000 | 1.0000000 | 1.0000000 |
| 1 | 0.057684 | 0.000103 | 0.9999941 | 0.9433906 |
| 2 | 0.120792 | 0.000213 | 0.9999743 | 0.8899735 |
| 3 | 0.189153 | 0.000331 | 0.9999374 | 0.8395667 |
| 4 | 0.262348 | 0.000457 | 0.9998801 | 0.7919987 |
| 5 | 0.339656 | 0.000592 | 0.9997989 | 0.7471079 |
| 6 | 0.420004 | 0.000736 | 0.9996909 | 0.7047426 |
| 7 | 0.501937 | 0.000891 | 0.9995528 | 0.6647597 |
| 8 | 0.583624 | 0.001057 | 0.9993831 | 0.6270253 |
| 9 | 0.662908 | 0.001236 | 0.9991806 | 0.5914135 |
| 10 | 0.737429 | 0.001428 | 0.9989470 | 0.5578068 |
| 11 | 0.804821 | 0.001635 | 0.9986841 | 0.5260943 |
| 12 | 0.862981 | 0.001858 | 0.9983966 | 0.4961725 |
| 13 | 0.910384 | 0.002098 | 0.9980900 | 0.4679435 |
| 14 | 0.946379 | 0.002357 | 0.9977694 | 0.4413144 |
| 15 | 0.971384 | 0.002637 | 0.9974385 | 0.4161962 |
| 16 | 0.986877 | 0.002939 | 0.9970996 | 0.3925045 |
| 17 | 0.995126 | 0.003265 | 0.9967509 | 0.3701578 |
| 18 | 0.998683 | 0.003617 | 0.9963878 | 0.3490783 |
| 19 | 0.999799 | 0.003998 | 0.9960028 | 0.3291919 |
| total | | | | 12.14644 |

The EPV of benefits after 20 years is $0.9955900(1.06)^{-20} \times 16.1807 = 5.022969$, so total EPV is $12.14644 + 5.022969 = 17.16941$.

Net premium is $45000 \times 17.16941 = \$772,623.45$.

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| Year | P(Husband Dies) | P(Wife Dies) | P(Both die) | P(One dies) |
|---|---|---|---|---|
| 1 | 0.000180 | 0.001651 | 0.00000029718 | 0.001830994 |
| 2 | 0.000193 | 0.001785 | 0.00000035343 | 0.001977993 |
| 3 | 0.000208 | 0.001929 | 0.000000401232 | 0.002136992 |
| 4 | 0.000223 | 0.002085 | 0.000000464955 | 0.002307991 |
| 5 | 0.000240 | 0.002254 | 0.00000054096 | 0.002493989 |
| 6 | 0.000258 | 0.002435 | 0.00000062823 | 0.002692987 |
| 7 | 0.000278 | 0.002631 | 0.000000731418 | 0.002908985 |

Annual EPV from one dying: 0.0138451

Annual EPV from both dying 0.000002863602

$x = (1+i)^{\frac{1}{12}}$

$$y = \frac{23}{144}x^{11} + \frac{21}{144}x^{10} + \frac{19}{144}x^9 + \frac{17}{144}x^8 + \frac{15}{144}x^7 + \frac{13}{144}x^6 + \frac{11}{144}x^5 + \frac{9}{144}x^4 + \frac{7}{144}x^3 + \frac{5}{144}x^2 + \frac{3}{144}x + \frac{1}{144}$$

$$\begin{aligned}
y - xy &= \frac{1}{144} + \frac{2}{144}(x + x^2 + \cdots + x^{11}) - \frac{23}{144}x^{12} \\
&= \frac{1}{144} + \frac{2x}{144}\left(\frac{x^{11}-1}{x-1}\right) - \frac{23}{144}x^{12} \\
y &= \frac{1}{144(x-1)}\left(23x^{12} - 1 - 2\frac{x^{11}-1}{x-1}\right) \\
&= \frac{1}{12i^{(12)}}\left(23i + 22 - 2\frac{x^{11}-1}{x-1}\right)
\end{aligned}$$

$i^{(12)} = 0.039288$
$y = 1.024815$

$$A^{(12\ 1)}{}_{45,76:\overline{7}|} = 0.0138451\frac{i}{i^{(12)}} + 0.000002863602 \times 1.024815 = 0.01410006$$

$$A^{(12)}_{45,76:\overline{7}|} = 0.01410006 + 0.997090283582(1.04)^{-7} = 0.7718067$$

$$d^{(12)} = 12\left(1 - (1.04)^{-\frac{1}{12}}\right) = 0.03915669$$

This gives $\ddot{a}^{(12)}_{45,76:\overline{7}|} = \frac{1-0.7718067}{0.03915669} = 5.827696$
So the monthly premiums are $\frac{850000\times 0.01410006}{12\times 5.827696} = \$171.38$

## 9.6 A Model with Dependent Future Lifetimes

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(a)

| Year | P(Husband Dies) | P(Wife Survives) | P(Wife Survives to 49) | P(Husband Dies and Wife Survives to 49) |
|---|---|---|---|---|
| 1 | 0.004599 | 0.999900 | $\frac{9961.92}{9997.47}$ | 0.004582188 |
| 2 | 0.004984 | 0.999794 | $\frac{9961.92}{9994.72}$ | 0.004966621 |
| 3 | 0.005398 | 0.999681 | $\frac{9961.92}{9991.74}$ | 0.005380173 |
| 4 | 0.005844 | 0.999560 | $\frac{9961.92}{9988.49}$ | 0.00582589 |
| 5 | 0.006324 | 0.999432 | $\frac{9961.92}{9984.95}$ | 0.00630583 |
| 6 | 0.006840 | 0.999294 | $\frac{9961.92}{9981.10}$ | 0.006822036 |
| 7 | 0.007394 | 0.999148 | $\frac{9961.92}{9976.91}$ | 0.007376601 |
| 8 | 0.007989 | 0.998991 | $\frac{9961.92}{9972.34}$ | 0.0079726 |
| 9 | 0.008625 | 0.998823 | $\frac{9961.92}{9967.36}$ | 0.008610147 |
| $\geqslant 10$ | 0.942003 | 0.998644 | 1 | 0.9407256 |
| | | | | 0.9985677 |

(b)

$$\sum(ap_1+(1-a)p_2)(aq_1+(1-a)q_2)(ar_1+(1-a)r_2) = \sum a^3p_1q_1r_1 + \sum a^2(1-a)p_1q_1r_2 + \cdots + \sum(1-a)^3p_2q_2r_2$$

We have

$$\frac{1^3}{12^3}+\frac{2^3}{12^3}+\cdots+\frac{12^3}{12^3}=\frac{12^2\times 13^2}{4\times 12^3}=3.520833$$
$$\frac{1^3}{12^3}+\frac{2^3}{12^3}+\cdots+\frac{11^3}{12^3}=\frac{12^2\times 11^2}{4\times 12^3}=2.520833$$
$$\frac{1^2\times 11}{12^3}+\frac{2^2\times 11}{12^3}+\cdots+\frac{11^2\times 1}{12^3}=\frac{8\times 12\times 13\times 23-12^2\times 13^2}{4\times 12^3}=0.63194440.9930556$$

| Year | P(Husband Dies) | P(Wife Survives) | P(Wife Survives to 49) | P(Husband Dies and Wife Survives to 49) |
|---|---|---|---|---|
| 1 | 0.004599 | 0.999900 | $\frac{9961.92}{9997.47}$ | 0.002102952 |
| 2 | 0.004984 | 0.999794 | $\frac{9961.92}{9994.72}$ | 0.004758375 |
| 3 | 0.005398 | 0.999681 | $\frac{9961.92}{9991.74}$ | 0.005156153 |
| 4 | 0.005844 | 0.999560 | $\frac{9961.92}{9988.49}$ | 0.005584445 |
| 5 | 0.006324 | 0.999432 | $\frac{9961.92}{9984.95}$ | 0.006045845 |
| 6 | 0.006840 | 0.999294 | $\frac{9961.92}{9981.10}$ | 0.006542403 |
| 7 | 0.007394 | 0.999148 | $\frac{9961.92}{9976.91}$ | 0.007076186 |
| 8 | 0.007989 | 0.998991 | $\frac{9961.92}{9972.34}$ | 0.007649737 |
| 9 | 0.008625 | 0.998823 | $\frac{9961.92}{9967.36}$ | 0.008264774 |
| 10 | 0.009305 | 0.998644 | 1 | 0.008687655 |
| $\geqslant 10$ | 0.932698 | 0.998644 | 1 | 0.9314333 |
| | | | | 0.9933018 |

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By summing over times the husband dies, we calculate the following lifetable for the wife.

Suppose the mortality for the wife while the husband is alive is $q_1$, the mortality while the husband is dead is $q_2$, and the mortality of the husband is $q_3$. Conditional on the husband dying at time $t$ in the year, the probability that the wife dies during the year is

$$tq_1+(1-tq_1)\left(1-\frac{1-q_2}{1-tq_2}\right)=tq_1+(1-tq_1)\frac{(1-t)q_2}{1-tq_2}$$

The total probability that the wife dies during the year if the husband is alive at the start of the year is therefore

$$
\begin{aligned}
&(1-q_3)q_1 + \int_0^1 q_3\left(tq_1 + (1-tq_1)\frac{(1-t)q_2}{1-tq_2}\right) dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3\int_0^1 (1-t)\frac{1-tq_1}{1-tq_2}\, dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3\int_0^1 \frac{1-(1+q_1)t+q_1t^2}{1-tq_2}\, dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3\int_0^1 \frac{1+q_1+\frac{q_1}{q_2}}{q_2} - \frac{q_1}{q_2}t + \frac{1-\frac{1+q_1+\frac{q_1}{q_2}}{q_2}}{1-tq_2}\, dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3\left(\frac{1+q_1+\frac{q_1}{q_2}}{q_2} - \frac{q_1}{2q_2} + \left(1-\frac{1+q_1+\frac{q_1}{q_2}}{q_2}\right)\left[-\frac{\log(1-tq_2)}{q_2}\right]_0^1\right) \\
&= (1-q_3)q_1 + q_3\left(1+q_1+\frac{q_1}{q_2}\right) + q_3\left(1-\frac{1+q_1+\frac{q_1}{q_2}}{q_2}\right)(\log(1)-\log(1-q_2)) \\
&= (1-q_3)q_1 + q_3\left(1+q_1+\frac{q_1}{q_2}\right) - q_3\left(\frac{{q_2}^2-q_2-q_1q_2-q_1}{{q_2}^2}\right)\log(1-q_2) \\
&= q_1 + q_3\left(\frac{q_1+q_2}{q_2}\right) + q_3\left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)
\end{aligned}
$$

If the probability that the husband and wife are both alive at the start of the year is $p_1$ and the probability that the wife is alive, but the husband is dead is $p_2$, then the probability that both are alive at the end of the year is $p_1(1-q_1)(1-q_3)$. The probability that the wife is alive and the husband is dead at the end of the year is therefore:

$$
\begin{aligned}
&p_2(1-q_2) + p_1\left(1-\left(q_1 + q_3\left(\frac{q_1+q_2}{q_2}\right) + q_3\left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)\right) - (1-q_1)(1-q_3)\right) \\
&= p_2(1-q_2) + p_1\left(q_3(1-q_1) - \left(q_3\left(\frac{q_1+q_2}{q_2}\right) + q_3\left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)\right)\right) \\
&= p_2(1-q_2) - p_1q_3\left(\left(\left(q_1+\frac{q_1}{q_2}\right) + \left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)\right)\right)
\end{aligned}
$$

| Year | Both Alive | Husband Dead, Wife Alive | Wife Alive |
|---|---|---|---|
| 0 | 10000.00 | 0.00 | 10000.00 |
| 1 | 9953.01 | 45.99 | 9999.00 |
| 2 | 9902.13 | 95.80 | 9997.93 |
| 3 | 9847.05 | 149.74 | 9996.79 |
| 4 | 9787.45 | 208.11 | 9995.56 |
| 5 | 9722.99 | 271.24 | 9994.23 |
| 6 | 9653.31 | 339.49 | 9992.80 |
| 7 | 9578.01 | 413.22 | 9991.23 |
| 8 | 9496.70 | 492.84 | 9989.54 |
| 9 | 9408.95 | 578.74 | 9987.70 |
| 10 | 9314.33 | 671.35 | 9985.69 |

This gives

$$a_{39:\overline{10}|} = 8.431041$$
$$A_{39:\overline{10}|} = 0.001132052$$

So the annual premium is

$$\frac{200000 \times 0.001132052}{8.431041} = \$26.85$$

## 9.7 The Common Shock Model

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We have

$${}_tp_{25:56}^{(00)} = e^{-\int_0^t 0.000042+0.000000001(56+s)+0.000000002(25+s)+0.000001(25+s)^2+0.000002(56+s)^2 ds} = e^{-\int_0^t 0.003403106+0.000174003s+0.000003s^2 ds} =$$

So the probability is given by

$$\int_0^{10} e^{-(0.003403106t+0.0000870015t^2+0.000001t^3)}(0.000002(25+t)^2+0.000000002(56+t))e^{-0.000002\frac{(66^3-(56+t)^3)}{3}}\,dt$$
$$= \int_0^{10} e^{-(0.07458667-0.002868894t-0.0000249985t^2-0.000001t^3)}(0.001250112+0.000100002t+0.000002t^2)dt$$
$$= 0.01715084$$

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If the husband dies first after time $t$, then the present value (at time of husband's death) of the life annuity is

$$25000\int_0^\infty e^{-0.005s}e^{-0.04s}\,ds = \frac{25000}{0.045} = 555555.56$$

The total transition intensity out of state 0 is

$$(0.000001(56+t)^2+0.000000001(25+t))+(0.000002(25+t)^2+0.000000002(56+t))+0.000042$$
$$=0.000003t^2+0.000212003t+0.004428137$$

The expected value of the life annuity is therefore

$$555555.56\int_0^\infty (0.000002(25+t)^2+0.000000002(56+t))e^{-(0.000003t^2+0.000212003t+0.004428137)}e^{-0.04t}\,dt$$
$$=555555.56\int_0^\infty (0.000002(25$$

(Numerically integrated)
The expected present value of the premiums is

$$P\int_0^\infty e^{-(0.000003t^2+0.000212003t+0.004428137)}e^{-0.04t}\,dt$$
$$=P\int_0^\infty e^{-(0.000003t^2+0.040212003t+0.004428137)}\,dt \qquad =P\int_0^\infty e^{-(0.000003(t+6}$$
$$=Pe^{134.746}\int_0^\infty e^{-0.000003(t+6702.0005)^2}\,dt$$
$$=Pe^{134.746}\sqrt{\frac{\pi}{0.000003}}\left(1-\Phi(6702.0005\sqrt{0.000006})\right)$$
$$=24.66746P$$

We therefore get that

$$24.66746P=84251.58$$
$$P=\$3,415.49$$

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We have

$${}_tp_{xy}^{(00)}=e^{-\int_0^t 0.001002(y+s)+0.002001(x+s)+0.012\,ds}=e^{-((0.001002y+0.002001x+0.012)s+0.0015015s^2)}$$

Numerically integrating gives the following lifetable

| Year | Both alive | H alive W dead | H dead W alive | Both dead |
|---|---|---|---|---|
| 0 | 10000.00 | 0.00 | 0.00 | 0.00 |
| 1 | 8247.83 | 239.88 | 1332.93 | 179.36 |
| 2 | 6782.28 | 394.62 | 2365.22 | 457.88 |
| 3 | 5560.41 | 485.42 | 3144.19 | 809.98 |
| 4 | 4545.01 | 529.17 | 3711.21 | 1214.61 |
| 5 | 3703.89 | 539.19 | 4102.32 | 1654.61 |
| 6 | 3009.38 | 525.84 | 4348.71 | 2116.08 |
| 7 | 2437.76 | 497.08 | 4477.26 | 2587.90 |
| 8 | 1968.80 | 458.92 | 4511.01 | 3061.27 |
| 9 | 1585.29 | 415.81 | 4469.58 | 3529.31 |
| 10 | 1272.65 | 370.99 | 4369.61 | 3986.74 |
| 11 | 1018.61 | 326.71 | 4225.09 | 4429.59 |
| 12 | 812.84 | 284.47 | 4047.71 | 4854.98 |
| 13 | 646.68 | 245.24 | 3847.19 | 5260.88 |
| 14 | 512.95 | 209.54 | 3631.53 | 5645.98 |
| 15 | 405.66 | 177.59 | 3407.22 | 6009.54 |
| 16 | 319.84 | 149.39 | 3179.52 | 6351.25 |
| 17 | 251.43 | 124.80 | 2952.61 | 6671.16 |
| 18 | 197.05 | 103.58 | 2729.76 | 6969.60 |
| 19 | 153.97 | 85.45 | 2513.48 | 7247.10 |
| 20 | 119.95 | 70.09 | 2305.62 | 7504.34 |
| 21 | 93.17 | 57.17 | 2107.54 | 7742.12 |
| 22 | 72.15 | 46.39 | 1920.14 | 7961.32 |
| 23 | 55.70 | 37.45 | 1743.97 | 8162.88 |
| 24 | 42.87 | 30.09 | 1579.29 | 8347.75 |
| 25 | 32.90 | 24.06 | 1426.12 | 8516.91 |
| 26 | 25.18 | 19.15 | 1284.33 | 8671.34 |
| 27 | 19.20 | 15.17 | 1153.63 | 8812.00 |
| 28 | 14.61 | 11.97 | 1033.61 | 8939.81 |
| 29 | 11.08 | 9.40 | 923.82 | 9055.71 |
| 30 | 8.37 | 7.35 | 823.71 | 9160.56 |
| 31 | 6.31 | 5.73 | 732.75 | 9255.21 |
| 32 | 4.74 | 4.44 | 650.34 | 9340.48 |
| 33 | 3.55 | 3.43 | 575.90 | 9417.11 |
| 34 | 2.65 | 2.64 | 508.86 | 9485.85 |
| 35 | 1.98 | 2.03 | 448.64 | 9547.36 |
| 36 | 1.47 | 1.55 | 394.69 | 9602.29 |
| 37 | 1.09 | 1.18 | 346.50 | 9651.24 |
| 38 | 0.80 | 0.89 | 303.54 | 9694.76 |
| 39 | 0.59 | 0.67 | 265.36 | 9733.38 |
| 40 | 0.43 | 0.51 | 231.49 | 9767.57 |
| 41 | 0.32 | 0.38 | 201.53 | 9797.77 |
| 42 | 0.23 | 0.28 | 175.08 | 9824.40 |
| 43 | 0.17 | 0.21 | 151.80 | 9847.82 |
| 44 | 0.12 | 0.16 | 131.34 | 9868.38 |

| Year | Both alive | H alive W dead | H dead W alive | Both dead |
|---|---|---|---|---|
| 45 | 0.09 | 0.12 | 113.41 | 9886.39 |
| 46 | 0.06 | 0.09 | 97.73 | 9902.13 |
| 47 | 0.05 | 0.06 | 84.04 | 9915.85 |
| 48 | 0.03 | 0.05 | 72.13 | 9927.79 |
| 49 | 0.02 | 0.03 | 61.78 | 9938.17 |
| 50 | 0.02 | 0.02 | 52.81 | 9947.15 |
| 51 | 0.01 | 0.02 | 45.05 | 9954.92 |
| 52 | 0.01 | 0.01 | 38.35 | 9961.63 |
| 53 | 0.01 | 0.01 | 32.58 | 9967.40 |
| 54 | 0.00 | 0.01 | 27.63 | 9972.36 |
| 55 | 0.00 | 0.00 | 23.38 | 9976.61 |
| 56 | 0.00 | 0.00 | 19.74 | 9980.25 |
| 57 | 0.00 | 0.00 | 16.64 | 9983.35 |
| 58 | 0.00 | 0.00 | 14.00 | 9986.00 |
| 59 | 0.00 | 0.00 | 11.75 | 9988.25 |
| 60 | 0.00 | 0.00 | 9.84 | 9990.15 |
| 61 | 0.00 | 0.00 | 8.23 | 9991.77 |
| 62 | 0.00 | 0.00 | 6.87 | 9993.13 |
| 63 | 0.00 | 0.00 | 5.72 | 9994.28 |
| 64 | 0.00 | 0.00 | 4.75 | 9995.25 |
| 65 | 0.00 | 0.00 | 3.94 | 9996.06 |
| 66 | 0.00 | 0.00 | 3.26 | 9996.74 |
| 67 | 0.00 | 0.00 | 2.70 | 9997.30 |
| 68 | 0.00 | 0.00 | 2.22 | 9997.78 |
| 69 | 0.00 | 0.00 | 1.83 | 9998.17 |
| 70 | 0.00 | 0.00 | 1.50 | 9998.50 |
| 71 | 0.00 | 0.00 | 1.23 | 9998.77 |
| 72 | 0.00 | 0.00 | 1.01 | 9998.99 |
| 73 | 0.00 | 0.00 | 0.82 | 9999.18 |
| 74 | 0.00 | 0.00 | 0.67 | 9999.33 |
| 75 | 0.00 | 0.00 | 0.54 | 9999.46 |
| 76 | 0.00 | 0.00 | 0.44 | 9999.56 |
| 77 | 0.00 | 0.00 | 0.36 | 9999.64 |
| 78 | 0.00 | 0.00 | 0.29 | 9999.71 |
| 79 | 0.00 | 0.00 | 0.23 | 9999.77 |
| 80 | 0.00 | 0.00 | 0.19 | 9999.81 |
| 81 | 0.00 | 0.00 | 0.15 | 9999.85 |
| 82 | 0.00 | 0.00 | 0.12 | 9999.88 |
| 83 | 0.00 | 0.00 | 0.10 | 9999.90 |
| 84 | 0.00 | 0.00 | 0.08 | 9999.92 |
| 85 | 0.00 | 0.00 | 0.06 | 9999.94 |
| 86 | 0.00 | 0.00 | 0.05 | 9999.95 |
| 87 | 0.00 | 0.00 | 0.04 | 9999.96 |
| 88 | 0.00 | 0.00 | 0.03 | 9999.97 |
| 89 | 0.00 | 0.00 | 0.02 | 9999.98 |

(a) Summing up we get that $A_{\overline{75,29}} = 0.4811416$. This gives $a_{\overline{75,29}} = \frac{1-0.4811416}{0.06} = 8.64764$. so the premium is $300000 \times 0.48114168.64764 = \$16,691.55$.

(b) After 10 years, if the husband is dead, but the wife is alive, then the mortality is $\mu_y^{23} = 0.002y$. The probability that she survives for $s$ years is therefore

$$e^{-\int_0^s 0.002(39+s)\,ds} = e^{-0.078s-0.001s^2}$$

This gives $A_{39} = \sum_{i=1}^{\infty}(1.06)^{-i}(e^{-0.078(i-1)-0.001(i-1)^2} - e^{-0.078i-0.001i^2}) = 0.590609$, so $\ddot{a}_{39} = \frac{1-0.590609}{0.06} = 6.823183$. This means that the policy value is

$$300000 \times 0.590609 - 16691.55 \times 6.823183 = \$63,293.20$$

## The Salary Scale Function

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(a) Average Salary from age 62–65 is given by $\frac{60000(1.03^{20}+1.03^{21}+1.03^{22})}{3} = \$111,650.18$

(b) Average Salary from age 62–65 is given by $\frac{60000(2.261+2.335+2.400)}{3} = \$139,920$

(c) for scale in (a) we have $\frac{60000(1.03^{19.6666667}+1.03^{20.6666667}+1.03^{21.6666667})}{3} = \$110,445.50$

for scale in (b), we can use linear interpolation to estimate $s_{42.33333333} = \frac{2}{3} \times 1 + \frac{1}{3} \times 1.036 = 1.012$. This gives a final average salary of $\frac{60000(2.261+2.335+2.400)}{3\times 1.012} = \$138,260.87$

## 10.4 Setting the DC Contribution

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If current salary is 1, final average salary is $\frac{1.03^{32}+1.03^{33}+1.03^{34}}{3} = 2.653108$. The replacement ratio means the original annuity is worth 1.591865, and the reversionary annuity is worth 0.7959323.

$${}_tp_{65} = e^{-\int_0^t 0.000002e^{\log(1.093)(65+t)}dt} = e^{-0.0006476502\left[\frac{e^{\log(1.093)s}}{\log(1.093)}\right]_0^t} = e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}$$

The value of the life annuity is given by

$$\overline{a}_{65} = \int_0^{\infty} e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 21.07607$$

The value of the reversionary annuity is given by

$$\overline{a}_{65|62} = \int_0^{\infty} \left(1 - e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right) e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 1.416998$$

So the EPV of the benefits at the time of retirement is

$$1.591865 \times 21.07607 + 0.7959323 \times 1.416998 = 34.67809$$

If first monthly salary is $x$, then we have

$$x(1+1.03^{\frac{1}{12}}+\ldots+1.03^{\frac{11}{12}})=1$$

$$x\frac{1.03-1}{1.03^{\frac{1}{12}}-1}=1$$

$$x=\frac{1.03^{\frac{1}{12}}-1}{0.03}=0.08220899$$

The accumulated value of all salary paid monthly in arrear at the end of 35 years is

$$0.08220899\sum_{i=1}^{420}(1.03)^{\frac{i}{12}}(1.06)^{35-\frac{i}{12}}=0.08220899(1.03)^{35}\sum_{i=1}^{420}\left(\frac{1.06}{1.03}\right)^{\frac{i}{12}}=0.08220899(1.03)^{35}\frac{\left(\frac{1.06}{1.03}\right)^{35}-1}{\left(\frac{1.06}{1.03}\right)^{\frac{1}{12}}-1}=167.2143$$

So the percentage of salary needed each month is

$$\frac{34.67809}{167.2143}=20.74\%$$

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For questions (a), (b), (e), (f) and (g), the employee's accumulated total is still 34.67809, and the final average salary is 2.653108.

For (a), the replacement ratio is $\frac{34.67809}{1.591865\times21.07607}\times60\%=62.02\%$

For (b), the value of the reversionary annuity is

$$\overline{a}_{65|62}=\int_0^\infty\left(1-e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right)e^{-0.01483447\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt=0.9445889$$

so the replacement ratio is $\frac{34.67809}{1.591865\times21.07607+0.7959323\times0.9445889}\times60\%=60.66\%$

For (e) the replacement ratio is $\frac{34.67809}{1.591865\times21.07607+0.4775595\times1.416998}\times60\%=60.79\%$

For (f) we have

$$\overline{a}_{65}=\int_0^\infty e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}e^{-log(1.03)t}dt=25.05707$$

The value of the reversionary annuity is given by

$$\overline{a}_{65|62}=\int_0^\infty\left(1-e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt=2.080705$$

so the replacement ratio is $\frac{34.67809}{1.591865\times25.05707+0.7959323\times2.080705}\times60\%=50.08\%$

For (g) we have

$$\overline{a}_{65}=\int_0^\infty e^{-0.08872485\left(e^{\log(1.143)t}-1\right)}e^{-log(1.04)t}dt=11.24322$$

and

$$\overline{a}_{65|62}=\int_0^\infty\left(1-e^{-0.08872485\left(e^{\log(1.143)t}-1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt=10.30211$$

so the replacement ratio is $\frac{34.67809}{1.591865\times11.24322+0.7959323\times10.30211}\times 60\% = 79.73\%$
For (c), the accumulated value of the investments is

$$0.08220899(1.03)^{35}\frac{\left(\frac{1.07}{1.03}\right)^{35}-1}{\frac{1.07}{1.03}-1} = 203.264$$

So the replacement ratio is $\frac{203.264}{167.2143}\times 60\% = 72.94\%$.
For (e), the accumulated value of the investments is

$$0.08220899(1.05)^{35}\frac{\left(\frac{1.06}{1.05}\right)^{35}-1}{\left(\frac{1.06}{1.05}\right)^{\frac{1}{12}}-1} = 225.7627$$

but the final average salary is $\frac{1.05^{32}+1.05^{33}+1.05^{34}}{3} = 5.007159$
So the new replacement ratio is $\frac{225.7627\times2.653108}{167.2143\times5.007159}\times 60\% = 42.92\%$.

## 10.5 The Service Table

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(a) See slide
(b) $\frac{62.28}{4882.68} = 0.01275529$

## 10.6 Valuation of Benefits

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(a)

$$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty}(1.05)^{-\frac{n}{12}}e^{-\int_0^{\frac{n}{12}} 0.0000023(1.12)^{65+t}\,dt} = \frac{1}{12}\sum_{n=0}^{\infty}(1.05)^{-\frac{n}{12}}e^{-0.0000023(1.12)^{65}\frac{(1.12)^{\frac{n}{12}}-1}{\log(1.12)}} = 14.18721$$

The member's final average salary is $76000\frac{(1.04)^{16}+(1.04)^{17}+(1.04)^{18}}{3} = \$148,116.36$.
The EPV of the accrued benefit conditional on the individual retiring at age 65 is therefore:

$$148116.36\times 13\times 0.01\times 14.18721(1.05)^{-18} = \$113,510.49$$

(b)

$$\ddot{a}_{60}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty}(1.05)^{-\frac{n}{12}}e^{-0.0000023(1.12)^{60}\frac{(1.12)^{\frac{n}{12}}-1}{\log(1.12)}} = 15.38250094$$

The member's final average salary is $76000\frac{(1.05)^{11}+(1.04)^{12}+(1.04)^{13}}{3} = \$121,740.85$.
The EPV of the accrued benefit conditional on the individual retiring at age 60 is therefore:

$$121740.85\times 13\times 0.01\times 15.38250094(1.05)^{-13} = \$129,105.80$$

(c)

| age | probability of retirement | EPV of benefits | Probability times EPV |
|---|---|---|---|
| 60 | 0.3 | 129105.80 | 38731.73853 |
| 60.5 | 0.09750441650 | 127563.3999 | 12437.99487 |
| 61.5 | 0.08392282902 | 124466.4712 | 10445.57838 |
| 62.5 | 0.07223304834 | 121353.9472 | 8765.765534 |
| 63.5 | 0.06217156087 | 118226.6896 | 7350.337829 |
| 64.5 | 0.05351155835 | 115085.7472 | 6158.417677 |
| 65 | 0.3306565869 | 113510.49 | 37532.9912 |
| total | | | 121422.824 |

So the EPV of accrued benefits is \$121,422.82.

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We calculate ${}_tp_{65} = e^{-\int_0^t 0.000002(1.102)^{65+t}\,dt} = e^{-0.000002(1.102)^{65}\left(\frac{1.102^t-1}{\log(1.102)}\right)} = e^{-0.01136305(1.102^t-1)}$

$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty} e^{-0.01136305(1.102^{\frac{n}{12}}-1)}(1.04^{-\frac{n}{12}} = 19.69323$

(a) If he withdraws today, he receives an annual pension of $75000 \times 0.02 \times 15 \times 1.02^{22} = \$34,784.54$

The EPV of this is $34784.54 \times 19.69323 \times_{22} p_{43}(1.04)^{-22}$

We have ${}_{22}p_{43} = e^{-\int_0^{22} 0.000002(1.102)^{43+t}\,dt} = e^{-0.000002(1.102)^{43}\left(\frac{1.102^{22}-1}{\log(1.102)}\right)} =$

So the EPV is $\$286,165.53$

(b) If the employee withdraws after t years,then his annual salary is $75000(1.05)^t$, so his accrued withdrawl benefits have present value $75000(1.05)^t \times 0.02 \times 15(1.02)^{22-t} \times 19.69323_{22-t}p_{43+t}(1.04)^{-22}$.

Conditional on the employee withdrawing before age 60, the EPV of the accrued withdrawl benefits is

$$\int_0^{17} \frac{286165.53}{{}_tp_{43}}\left(\frac{1.05}{1.02}\right)^t e^{-0.07(43+t)} e^{\frac{e^{-0.07t}-e^{-0.07times17}}{0.07}}\,dt = \$215,505.95$$

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We calculate ${}_tp_{65} = e^{-\int_0^t 0.0000023(1.12)^{65+t}\,dt} = e^{-0.0000023(1.12)^{65}\left(\frac{1.12^t-1}{\log(1.12)}\right)} = e^{-0.03210402(1.12^t-1)}$

$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty} e^{-0.03210402(1.12^{\frac{n}{12}}-1)}(1.05^{-\frac{n}{12}} = 14.03364$

If the member withdraws at age $x$, then the salary is $45000(1.04)^{x-46}\left(\frac{1.04^{-1}+1.04^{-2}+1.04^{-3}}{3}\right)$, so with the COLA, the accrued pension has an annual value of $41626.3655 \times 0.015 \times 13(1.04)^{x-46}(1.02)^{65-x} = 11825.14209\left(\frac{1.04}{1.02}\right)^{x-46}$, so the value at age 65 is

$$165949.787\left(\frac{1.04}{1.02}\right)^{x-46}$$

. Discounting at 5% to the age of withdrawl and at 6% to the present day, gives a conditional present value of

$$165949.787\left(\frac{1.04 \times 1.06}{1.02 \times 1.05}\right)^{x-46}(1.05)^{46-65} = 65671.96589\,(1.029318394)^{x-46}$$

The EPV is therefore $65671.96589\,(1.029318394)^{x-46}\,{}_{65-x}p_{46+x}$

We have ${}_{65-x}p_{46+x} = e^{-0.0000023(1.12)^{65}\left(\frac{1-1.12^{x-65}}{\log(1.12)}\right)} = e^{-0.03210402(1-1.12^{x-65})}$ so the EPV is $65671.96589\,(1.029318394)^{x-46}\,e^{-0.032}$

The probability that the member is still enrolled in the plan at age $x$ is $e^{-\int_{46}^x e^{-0.07y}+0.0000023(1.12)^y\,dy} = e^{-\frac{(e^{-3.22}-e^{-0.07x})}{0.07}-0.00002029494998(1.12^x-1.12^{46})}$ the EPV of accrued pension benefits paid to early withdrawls is therefore

$$\int_{46}^{60} 65671.96\,(1.02932)^{x-46}\, e^{-0.03210(1-1.12^{x-65})} e^{-0.07x} e^{-\frac{(e^{-3.22}-e^{-0.07x})}{0.07}-0.000020295(1.12^{x}-1.12^{46})}$$
$$= 26792.25309$$

The probability he is still employed at age 60 is $e^{-\int_{46}^{60} e^{-0.07x}+0.0000023(1.12)^x\,dx} = e^{-\frac{e^{-3.22}-e^{-4.2}}{0.07}-0.0000023\frac{1.12^{60}-1.12^{46}}{\log(1.12)}} = 0.6900072247$

If this happens, then his final average salary is $45000\left(\frac{1.04^{17}+1.04^{18}+1.04^{19}}{3}\right) = 91208.4928$

He has probability 0.3 of retiring at age 60, in which case the expected value of the accrued pension is $91208.4928 \times 13 \times 0.015\ddot{a}_{60}^{(12)} = 272954.8931$. The EPV of pension benefits from retirements at age 60 is therefore $272954.8931 \times 0.3 \times 0.6900072247(1.06)^{-14} = 24991.00164$.

To simplify, we assume the remaining retirements, except at age 65 happen in the middle of their year. We get the following:

| age | P(retire) | $\ddot{a}_x^{(12)}$ | $S_{\text{Fin}}$ | EPV(Pension Benefits) |
|---|---|---|---|---|
| 60.5 | 0.007984369979 | 15.22710298 | 93014.77692 | 947.3478018 |
| 61.5 | 0.007517395163 | 14.98002832 | 96735.3680 | 860.9124779 |
| 62.5 | 0.007113475961 | 14.72279172 | 100604.7827 | 785.5584389 |
| 63.5 | 0.006768439094 | 14.45514241 | 104628.9740 | 720.0204573 |
| 64.5 | 0.006478882979 | 14.17684657 | 108814.1330 | 663.1949110 |
| 65 | 0.4471424941 | 14.03363874 | 110969.0775 | 44878.8153482 |
| total | | | | 48855.84944 |

So the total EPV of accrued pension benefits is $48855.84944 + 26792.25309 + 24991.00164 = \$100,639.10$

## 10.7 Funding the Benefits

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(a) Under the projected unit method, the final average salary is expected to be $47000\left(\frac{1.05^{18}+1.05^{19}+1.05^{20}}{3}\right) = 118860.9184$

We have $\ddot{a}_{65}^{(12)} = 13.95103541$. Therefore the EPV of accrued benefits for an individual who reaches retirement age is $118860.9184 \times 0.02 \times 26 \times 13.95103541(1.04)^{-20} = 393533.8372$.

The probability that this individual reaches retirement age is ${}_{20}p_{45} = e^{-0.0000076\left(\frac{1.1087^{65}-1.1087^{45}}{\log(1.1087)}\right)} = 0.948743622$.

So the EPV of benefits is $0.948743622 \times 393533.8372 = 373362.7181$.

After another year, the projected final average salary will still be 113200.8747, so the EPV conditional on surviving to retirement age will be $118860.9184 \times 0.02 \times 27 \times 13.95103541(1.04)^{-19} = 425016.5442$.

The probability of surviving to retirement age is $e^{-0.0000076\left(\frac{1.1087^{65}-1.1087^{46}}{\log(1.1087)}\right)} = 0.9495331634$. The EPV of benefits at the end of the year is therefore $0.9495331634 \times 425016.5442 = 403567.3037$.

The accumulated value of the reserves at the begining of the year is $373362.7181(1.04) = 388297.2268$, so the annual contribution is $403567.3037 - 388297.2268 = \$15,270.08$.

(b) Under the traditional unit method, the final average salary is $47000\left(\frac{1.05^{-2}+1.05^{-1}+1}{3}\right) = 44797.43008$

The value in the current year is therefore $44797.43008 \times 0.02 \times 26 \times 13.95103541(1.04)^{-20}{}_{20}p_{45} = 140716.4818$

If the member survives the year, the final average salary in one year's time is $47000\left(\frac{1.05^{-1}+1+1.05}{3}\right) = 47037.30159$ so EPV at the end of next year is $47037.30159 \times 0.02 \times 27 \times 13.95103541(1.04)^{-20}{}_{20}p_{45} = 159705.2861$.

The accumulated value of the assets funding the benefit at the start of the year is $140716.4818(1.04) = 146345.1411$, so the contribution is $159705.2861 - 146345.1411 = \$13,360.14$.

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The rate of exit (for ages below 60) is $\mu_x^{01} + \mu_x^{02} = 0.2e^{-0.04x} + 0.00000187(1.130)^x$ so the probability of the employee remaining employed at age $x$ is

$$e^{-\int_{46}^{x} 0.2e^{-0.04t}+0.00000187(1.130)^t\,dt} = e^{-\left[\frac{0.00000187}{\log(1.130)}(1.130)^t - \frac{0.2}{0.04}e^{-0.04t}\right]_{46}^{x}} = e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)}$$

If the individual has retired, is at age $x$ and has passed the guaranteed time of the pension, the value of the pension at that time is $R\sum_{i=0}^{\infty} {}_ip_x 1.05^{-i}$ where $R$ is the regular pension payment. We have that

$${}_tp_x = e^{-0.00000187(1.13)^x \int_0^t e^{s\log(1.13)}\,ds} = e^{-0.00000187(1.13)^x\left(\frac{(1.13)^t-1}{\log(1.13)}\right)}$$

This gives that the value of the pension is

$$R\sum_{i=0}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^i-1}{\log(1.13)}\right)} 1.05^{-i}$$

If the individual retires at age $x$, with pension value $R$ then the present value of the pension at that time is

$$R\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^i-1}{\log(1.13)}\right)} 1.05^{-i}\right)$$

and the present value is

$$(1.05)^{-19}R\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)} 1.05^{-i}\right)$$

If the member exits at age $x$, then the final average salary is $87000(1.06)^{x-46}\left(\frac{1+(1.06)^{-1}+(1.06)^{-2}}{3}\right) = 82168.39(1.06)^{x-46}$.

If the member withdraws at age $x$, then the final average salary is also $82168.39(1.06)^{x-46}$, and with COLA, the eventual accrued pension benefit is $82168.39(1.06)^{x-46}(1.02)^{65-x} = 20400.75(1.039216)^x$ per 100 years of service.

Since the employee has 12 years of service, the accrued pension beneifit if he withdraws at age $x$ is

$$2448.090(1.039216)^x$$

The overall expected pension benefit for an individual who withdraws is therefore

$$\begin{aligned}&\int_{46}^{60} 0.2e^{-0.04x} e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)} \times 2448.090(1.039216)^x \times \\ &\qquad (1.05)^{-19}\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)} 1.05^{-i}\right) dx \\ &= 27393.66\end{aligned}$$

After another year, the expected pension benefit to an individual who withdraws will be

$$\int_{47}^{60} 0.2e^{-0.04x}e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)} \times 2448.090(1.039216)^x \times$$
$$(1.05)^{-19}\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)} 1.05^{-i}\right)\,dx$$
$$= 25057.95$$

In addition, annother year will have accrued, so the expected benefit is $\frac{13}{12} \times 25057.95 = 27146.11$. (Valued at the present time, not at age 47.)

If the individual is still employed at age 60, and retires at age $x$, then the present value of the accrued pension is

$$0.12 \times 87000(1 + 1.06^{-1} + 1.06^{-2})1.06^{x-46}1.05^{46-x}\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i}-1}{\log(1.13)}\right)} 1.05^{-i}\right)$$

Denote this value by $P(x)$. We then have that the expected pension payments to individuals who retire are

$$ {}_{14}p_{46}^{(00)}\left(0.08P(60) + 0.92\int_0^5 0.1e^{-0.1t-0.00000187(1.13)^{60}\left(\frac{(1.13)^t-1}{\log(1.13)}\right)}P(60+t)\,dt + {}_{19}p_{46}P(65)\right) = 12053.66\,{}_{14}p_{46}^{(00)}$$

We have that

$$ {}_{14}p_{46}^{(00)} = e^{-0.004229764\left((1.130)^{14}-1\right)-0.7940871\left(1-e^{-0.04(14)}\right)} = 0.6979008$$

So the expected payments for individuals who retire are $0.6979008 \times 12053.66 = \$8412.26$.

The expected payments conditional on being employed at the end of the year are

$$12053.66\,{}_{13}p_{47}^{(00)} = 12053.66 \times 0.7113312 = 8574.145$$

.

Multiplying by $\frac{13}{12}$ to account for the additional year of service, we get $\frac{13}{12} \times 8574.145 = 9288.66$.

Finally, conditional on the individual surviving for the year, the expected death benefits decrease by the expected death benefits for the year, which is

$$\int_0^1 0.00000187(1.13)^{46+t}e^{-\frac{0.00000187}{\log(1.13)}(1.13)^{46}(1.13^t-1)-5e^{-1.84}(1-e^{-0.04t})} \times 3 \times 87000(1.06^t)\,dt = 145.47$$

The increase in the present value of expected benefits is therefore

$$27146.11 - 27393.66 + 9288.66 - 8412.26 - 145.47 = \$483.38$$

The employee contribution is 4% of 87000, which is \$3,480.

## 11.2 The Yield Curve

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$\left(\frac{1.048^6}{1.043^3}\right)^{\frac{1}{3}} = \frac{1.048^2}{1.043} = 0.05302396932$, so the annual forward rate is 5.30%.

## 11.3 Valuation of Insurances and Life Annuities

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See slide. $\frac{300000\times0.001794442}{4.638626} = 116.05$.

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See slide.

## 11.4 Diversifiable and Non-diversifiable Risks

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(a) If the interest rate changes to 0.04 after 1 year, then the expected accumulated value of the premiums received at the end of the term is

$$91.95\left((1.05)(1.04)^9 + 0.999977(1.04)^9 + 0.999952(1.04)^8 + 0.999926(1.04)^7 + 0.999898(1.04)^6 + 0.999868(1.04)^5 + 0.999837(1.04)^4 +$$

The expected accumulated value of the death benefits is

$$3200000(0.000023(1.04)^9+0.000025(1.04)^8+0.000026(1.04)^7+0.000028(1.04)^6+0.000029(1.04)^5+0.000031(1.04)^4+0.000033(1.04)^3+0$$

So the expected loss is $1160.44 - 1149.30 = \$11.15$

The present value of this loss is $11.15(1.04)^{-9}(1.05)^{-1} = 7.46$.

(b)

If the interest rate changes to 0.06 after 1 year, then the expected accumulated value of the premiums received at the end of the term is

$$91.625\left((1.05)(1.06)^9 + 0.999977(1.06)^9 + 0.999952(1.06)^8 + 0.999926(1.06)^7 + 0.999898(1.06)^6 + 0.999868(1.06)^5 + 0.999837(1.06)^4$$

The expected accumulated value of the death benefits is

$$3200000(0.000023(1.06)^9+0.000025(1.06)^8+0.000026(1.06)^7+0.000028(1.06)^6+0.000029(1.06)^5+0.000031(1.06)^4+0.000033(1.06)^3+0$$

So the expected profit is $1283.00 - 1261.99 = \$21.01$.

The present value of this profit is $21.01(1.06)^{-9}(1.05)^{-1} = 11.84$.

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If the interest rate in 1 years time is $i = 0.05$, then the variance of the present value of aggregate loss on $N$ policies is

conditional on dying:

$$\mathbb{E}(L) = (3200000+21\times91.95)\left(\frac{0.23(1.05)^{-1} + 0.25(1.05)^{-2} + 0.26(1.05)^{-3} + 0.28(1.05)^{-4} + 0.29(1.05)^{-5} + 0.31(1.05)^{-6} + 0.33(1.05}{3.09}\right.$$

and

$$\mathbb{E}((L-21\times 91.95)^2) = (3200000+21\times 91.95)^2 \left(\frac{0.23(1.05)^{-2} + 0.25(1.05)^{-4} + 0.26(1.05)^{-6} + 0.28(1.05)^{-8} + 0.29(1.05)^{-10} + 0.31(1.0}{3.09}\right.$$

So

$$\mathrm{Var}(L) = 5919637281997 - 2405327.962^2 = 134034677903$$

Conditional on surviving

$$E(L) = -91.95\left(\frac{1.05 - 1.05^{-9}}{0.05}\right) = -745.5142031$$

and

$$\mathrm{Var}(L) = 0$$

Overall, we get

$$\mathrm{Var}(L) = 134034677903 \times 0.000309 + 0 + (2403397.012 + 745.5142031)^2 \times 0.999691 \times 0.000309 = 1826854342$$

For the aggregate loss on $N$ policies, we have

$$E(L) = -2.634162523N$$

$$\mathrm{Var}(L) = 1826854342N$$

Similarly, if interest rates fall to 1.04 in one year, PV future loss has

$$E(L) = 7.456161644N$$

$$\mathrm{Var}(L) = 1982460823N$$

If interest rates increase to 1.06 in one year:

$$E(L) = -11.84360483N$$

$$\mathrm{Var}(L) = 1689773860N$$

The overall variance of the aggregate loss is therefore

$$(0.2\times 1689773860 + 0.6\times 1826854342 + 0.2\times 1982460823)N + 37.29465577N^2 = 1830559542N + 37.29465577N^2$$

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The premium is $200000(1.05)^{-1} \times 0.00032 = \$60.95$.

We have the following:

| $q_{58}$ | probability | $\mathbb{E}(L)$ | $\mathrm{Var}(L)$ |
|---|---|---|---|
| 0.00028 | 0.26 | −7.616666667 | 10155885.71 |
| 0.00032 | 0.58 | 0.00238095 | 11606262.13 |
| 0.00033 | 0.14 | 1.907142857 | 11968838.10 |
| 0.00077 | 0.02 | 85.71666667 | 27914996.83 |

So the variance of the aggregate loss for $N$ policies is given by:

$0.26 \times 10155885.71 + 0.58 \times 11606262.13 + 0.14 \times 11968838.10 + 0.02 \times 27914996.83 = 11606099.59$

$1.907142857 \times 0.14 + 85.71666667 \times 0.02 - 7.616666667 \times 0.26 + 0.00238095 \times 0.58 = 0.00238095096$

$1.907142857^2 \times 0.14 + 85.71666667^2 \times 0.02 + 7.616666667^2 \times 0.26 + 0.00238095^2 \times 0.58 - 0.00238095096^2 = 162.5396825$

So the aggregate variance is $11606099.59N + 162.5396825N^2$.

## 11.5 Monte Carlo Simulation

40

Probability of surviving $t$ years is $e^{-\int_{47}^{47+t} 0.0000164(1.088)^x\,dx} = e^{-\frac{0.0000164(1.088)^{47}}{\log(1.088)}((1.088)^t-1)} = e^{-0.01024147(1.088^t-1)}$, so for the simulations

$$\begin{aligned}
e^{-0.01024147(1.088^t-1)} &= u\\
0.01024147(1.088^t-1) &= -\log(u)\\
1.088^t &= 1-\frac{\log(u)}{0.01024147}\\
t &= \frac{\log\left(1-\frac{\log(u)}{0.01024147}\right)}{\log(1.088)}
\end{aligned}$$

For the interest rate, we have that $\log(i) - \log(0.04)$ is normally distributed with mean 0 and standard deviation 0.4.

$$i = 0.04e^{0.4\Phi^{-1}(v)}$$

We therefore have the following:

| No. | $u$ | $t$ | $v$ | $i$ | Premiums | Benefits | PVFL |
|---|---|---|---|---|---|---|---|
| 1 | 0.6129116 | 53.8 | 0.4397 | 0.03764344 | 199912.78 | 256717.52 | 56804.74 |
| 2 | 0.6120158 | 53.8 | 0.2549 | 0.03073115 | 199912.78 | 281238.80 | 81326.02 |
| 3 | 0.9504287 | 67.4 | 0.8275 | 0.05835981 | 199912.78 | 218301.91 | 18389.13 |

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Standard deviation of simulation mean is $\frac{64278.49}{100} = 642.7849$. Using a normal approximation, a 95% confidence interval is 1.96 standard deviations to either side of the mean. That is

$$[176.4161 - 642.7849 \times 1.96, 176.4161 + 642.7849 \times 1.96] = [-1083.44, 1436.27]$$

## 12.3 Profit Testing a Term Insurance Policy

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See slide.

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See slide.

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See slide.

## 12.5 Profit Measures

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See slide.

## 12.6 Using the Profit Test to Calculate the Premium

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At a risk discount rate of 10%, the NPV is

$$P\left((1.04)(1.1)^{-1}+0.9996\left(\frac{1.1^{-1}-1.1^{-10}}{0.1}\right)\right)-(160+56.34(1.1)^{-1}+59.24(1.1)^{-2}+62.50(1.1)^{-3}+66.13(1.1)^{-4}+69.93(1.1)^{-5}+74.11($$

The premium should be chosen to make this equal to zero. That is

$$\begin{aligned}6.178837P-590.908&=0\\6.178837P&=590.908\\P&=\$95.63\end{aligned}$$

## 12.7 Using the Profit Test to Calculate Reserves

47

Recalling Question 42, with a premium of \$90, The net outflows without reserves are all positive except for the final year, where the net outflow is $-4.61$. To correct for this, the reserve $R$ must satisfy $1.04R = 4.61$. This means $R = \$4.43$. This reserve is a negative cash flow at the end of year 9, so the net cash flow from year 9 is $1.03-4.43 = -3.40$. To prevent this negative cash flow, the reserve needs to satisfy $1.04R = 3.40$, so $R = 3.27$. This is a negative cash flow at the end of year 8, so the net cash-flow in that year is $6.48 - 3.27 = 3.21$, which is positive, so reserves of \$3.27 in year 9 and \$4.43 in year 10 are needed.

## 12.8 Profit Testing for Multiple-State Models

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Recall from Question 48 that the probabilities of the life being in each state are:

| $t$ | ${}_tp_{37}^{00}$ | ${}_tp_{37}^{01}$ | ${}_tp_{37}^{02}$ |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0.99812 | 0.000375 | 0.001505 |
| 2 | 0.99617 | 0.000750 | 0.003083 |
| 3 | 0.99414 | 0.001127 | 0.004736 |
| 4 | 0.99203 | 0.001505 | 0.006464 |
| 5 | 0.98985 | 0.001884 | 0.008271 |
| 6 | 0.98758 | 0.002263 | 0.010156 |
| 7 | 0.98523 | 0.002644 | 0.012123 |
| 8 | 0.98280 | 0.003025 | 0.014171 |
| 9 | 0.98029 | 0.003407 | 0.016303 |
| 10 | 0.97769 | 0.003790 | 0.018519 |

If the life is in the healthy state at the start of year $i$, and is then aged $x$, the probability of being sick at the end is

$$e^{0.000001\begin{pmatrix} -\left(1881+\frac{1}{3}+77x+x^2\right) & 375+2x & 1506+\frac{1}{3}+75x+x^2 \\ 67.5+x & -(342.5+3x) & 275+2x \\ 0 & 0 & 0 \end{pmatrix}}$$

Probabilities of being sick or dead at the end of each year for a life alive at start of year are

| $t$ | $p_{37+t}^{00}$ | $p_{37+t}^{01}$ | $p_{37+t}^{02}$ | $p_{37+t}^{10}$ | $p_{37+t}^{11}$ | $p_{37+t}^{12}$ |
|---|---|---|---|---|---|---|
| 0 | 0.9981204 | 0.0003745833 | 0.001504969 | 0.00006742499 | 0.9996576 | 0.0002750037 |
| 1 | 0.9980426 | 0.0003765658 | 0.001580836 | 0.00006842111 | 0.9996546 | 0.0002770063 |
| 2 | 0.9979628 | 0.0003785478 | 0.001658694 | 0.00006941708 | 0.9996516 | 0.0002790090 |
| 3 | 0.9978809 | 0.0003805293 | 0.001738542 | 0.00007041290 | 0.9996486 | 0.0002810119 |
| 4 | 0.9977971 | 0.0003825102 | 0.001820380 | 0.00007140856 | 0.9996456 | 0.0002830149 |
| 5 | 0.9977113 | 0.0003844905 | 0.001904206 | 0.00007240405 | 0.9996426 | 0.0002850181 |
| 6 | 0.9976235 | 0.0003864703 | 0.001990020 | 0.00007339939 | 0.9996396 | 0.0002870214 |
| 7 | 0.9975337 | 0.0003884495 | 0.002077823 | 0.00007439457 | 0.9996366 | 0.0002890249 |
| 8 | 0.9974420 | 0.0003904281 | 0.002167613 | 0.00007538957 | 0.9996336 | 0.0002910285 |
| 9 | 0.9973482 | 0.0003924062 | 0.002259390 | 0.00007638440 | 0.9996306 | 0.0002930323 |

| $t$ | Premium | Exp | Interest | Expected Disability Benefit | Expected Death Benefit | Net Cash Flow |
|---|---|---|---|---|---|---|
| 0 | | 200 | | | | −200 |
| 1 | 489.45 | | 34.26 | 29.96666 | 300.9938 | 192.75108 |
| 2 | 489.45 | | 34.26 | 30.12527 | 316.1673 | 177.41896 |
| 3 | 489.45 | | 34.26 | 30.28383 | 331.7389 | 161.68881 |
| 4 | 489.45 | | 34.26 | 30.44234 | 347.7084 | 145.56071 |
| 5 | 489.45 | | 34.26 | 30.60081 | 364.0759 | 129.03478 |
| 6 | 489.45 | | 34.26 | 30.75924 | 380.8412 | 112.11110 |
| 7 | 489.45 | | 34.26 | 30.91762 | 398.0041 | 94.78978 |
| 8 | 489.45 | | 34.26 | 31.07596 | 415.5646 | 77.07092 |
| 9 | 489.45 | | 34.26 | 31.23425 | 433.5226 | 58.95464 |
| 10 | 489.45 | | 34.26 | 31.39249 | 451.8780 | 40.44103 |

| $t$ | Premium | Exp | Interest | Expected Disability Benefit | Expected Death Benefit | Net Cash Flow |
|---|---|---|---|---|---|---|
| 0 | | 200 | | | | $-200$ |
| 1 | 489.45 | | 34.26 | 79972.61 | 55.00074 | $-79503.89$ |
| 2 | 489.45 | | 34.26 | 79972.37 | 55.40126 | $-79504.06$ |
| 3 | 489.45 | | 34.26 | 79972.13 | 55.80181 | $-79504.22$ |
| 4 | 489.45 | | 34.26 | 79971.89 | 56.20238 | $-79504.38$ |
| 5 | 489.45 | | 34.26 | 79971.65 | 56.60299 | $-79504.54$ |
| 6 | 489.45 | | 34.26 | 79971.41 | 57.00362 | $-79504.70$ |
| 7 | 489.45 | | 34.26 | 79971.17 | 57.40429 | $-79504.86$ |
| 8 | 489.45 | | 34.26 | 79970.93 | 57.80498 | $-79505.02$ |
| 9 | 489.45 | | 34.26 | 79970.69 | 58.20571 | $-79505.18$ |
| 10 | 489.45 | | 34.26 | 79970.45 | 58.60646 | $-79505.34$ |

We see that expected cash flows are all negative if the life starts the year in the sick state, and positive if the life starts the year in the healthy state. We need to calculate separate reserves for each state in the usual way by working backwards.

For the 10th year, if the life is in the sick state, the expected net cash flow is $-\$79,505.34$, so the reserve needed is $79505.34(1.07)^{-1} = 74304.06$.

For a life in the sick state at the start of year 9, the probability that the life is in the sick state at the end of year 9 is 0.9996306, so the expected reserves needed are $0.9996306 \times 74304.06 = \$74276.61$. For a life in the healthy state at the start of year 9, the probability of being in the sick state at the end is 0.0003924, so the additional expected cashflow is $-0.0003924 \times 74304.06 = -\$29.15691$, making the total expected net cash flow for that year \$29.80.

We proceed up the table in this way.

| $t$ | Reserve | Premium | Exp | Interest | Expected Disability Benefit | Expected Death Benefit | Expected Reserve | Net Cash Flow |
|---|---|---|---|---|---|---|---|---|
| 2 | 530748.32 | 489.45 | | 34.26 | 79972.37 | 55.40126 | 488396.64 | $-567900.70$ |
| 3 | 488566.86 | 489.45 | | 34.26 | 79972.13 | 55.80181 | 443262.32 | $-522766.54$ |
| 4 | 443418.14 | 489.45 | | 34.26 | 79971.89 | 56.20238 | 394953.03 | $-474457.41$ |
| 5 | 395093.05 | 489.45 | | 34.26 | 79971.65 | 56.60299 | 343245.02 | $-422749.56$ |
| 6 | 343367.74 | 489.45 | | 34.26 | 79971.41 | 57.00362 | 269064.28 | $-348568.98$ |
| 7 | 269161.28 | 489.45 | | 34.26 | 79971.17 | 57.40429 | 208497.72 | $-288002.58$ |
| 8 | 208573.51 | 489.45 | | 34.26 | 79970.93 | 57.80498 | 143668.64 | $-223173.66$ |
| 9 | 143721.30 | 489.45 | | 34.26 | 79970.69 | 58.20571 | 74276.61 | $-153781.79$ |
| 10 | 74276.61 | 489.45 | | 5235.55 | 79970.45 | 58.60646 | 0 | $-79505.34$ |

There is no need to calculate the reserves for the first year, since we know that the life is in the healthy state at the start of year 1.

Knowing the reserves needed if the life is in the sick state, we can calculate the reserves as extra expenses in the healthy state:

| $t$ | Reserve | Premium | Exp | Interest | Expected Disability Benefit | Expected Death Benefit | Expected Reserve Sick | Expected Reserve Healthy | Net Cash Flow |
|---|---|---|---|---|---|---|---|---|---|
| 0 | | | 200 | | | | | 21.65 | −221.65 |
| 1 | 21.65 | 489.45 | | 34.26 | 29.96666 | 300.9938 | 198.81 | 17.11 | −23.17 |
| 2 | 17.14 | 489.45 | | 34.26 | 30.12527 | 316.1673 | 183.98 | 11.78 | −18.34 |
| 3 | 11.80 | 489.45 | | 34.26 | 30.28383 | 331.7389 | 167.85 | 6.47 | −12.63 |
| 4 | 6.49 | 489.45 | | 34.26 | 30.44234 | 347.7084 | 150.34 | 2.15 | −6.94 |
| 5 | 2.16 | 489.45 | | 34.26 | 30.60081 | 364.0759 | 131.34 | | −2.31 |
| 6 | | 489.45 | | 34.26 | 30.75924 | 380.8412 | 103.49 | | 8.62 |
| 7 | | 489.45 | | 34.26 | 30.91762 | 398.0041 | 80.61 | | 14.18 |
| 8 | | 489.45 | | 34.26 | 31.07596 | 415.5646 | 55.83 | | 21.24 |
| 9 | | 489.45 | | 34.26 | 31.23425 | 433.5226 | 29.00 | | 29.95 |
| 10 | | 489.45 | | 34.26 | 31.39249 | 451.8780 | | | 40.44 |

The resulting profit signature is then

$$
\begin{aligned}
-221.65 \times 1 &= -221.65 \\
0 \times 1 &= 0 \\
0 \times 0.99812 &= 0 \\
0 \times 0.99617 &= 0 \\
0 \times 0.99414 &= 0 \\
0 \times 0.99203 &= 0 \\
8.62 \times 0.98985 &= 8.53 \\
14.18 \times 0.98758 &= 14.00 \\
21.24 \times 0.98523 &= 20.93 \\
29.95 \times 0.98280 &= 29.43 \\
40.44 \times 0.98029 &= 39.64
\end{aligned}
$$

## 13.4 Universal Life Insurance

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See slide.

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Profit signature is given by

| Year | Profit vector | Probability in force | Profit signature |
|---|---|---|---|
| 1 | 257.6443 | 1.0000000 | 257.6443 |
| 2 | 251.8084 | 0.9489626 | 238.9567 |
| 3 | 276.0940 | 0.9004360 | 248.6050 |
| 4 | 349.0236 | 0.8812669 | 307.5830 |
| 5 | 423.4032 | 0.8623908 | 365.1390 |
| 6 | 501.1323 | 0.8437907 | 422.8508 |
| 7 | 583.2349 | 0.8254506 | 481.4316 |
| 8 | 654.6027 | 0.7991134 | 523.1018 |
| 9 | 694.4401 | 0.7734466 | 537.1123 |
| 10 | 700.2219 | 0.7484175 | 524.0583 |
| 11 | 704.9742 | 0.7239944 | 510.3974 |
| 12 | 708.5417 | 0.7001450 | 496.0820 |
| 13 | 710.7165 | 0.6768365 | 481.0389 |
| 14 | 711.2783 | 0.6472935 | 460.4058 |
| 15 | 709.9592 | 0.6187523 | 439.2889 |

At the risk discount rate the NPV is

$$257.64(1.1)^{-1}+238.96(1.1)^{-2}+248.61(1.1)^{-3}+307.58(1.1)^{-4}+365.14(1.1)^{-5}+422.85(1.1)^{-6}+481.43(1.1)^{-7}+523.10(1.1)^{-8}+537.11(1$$

The EPV of the premiums received is

$$3160(1+0.9489626(1.1)^{-1}+0.9004360(1.1)^{-2}+0.8812669(1.1)^{-3}+0.8623908(1.1)^{-4}+0.8437907(1.1)^{-5}+0.8254506(1.1)^{-6}+0.799113$$

The profit margin is $\frac{1117.60}{16615.07} = 0.067$

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We have

$$\begin{aligned}
CoI_t &= q_t(1.06)^{-1}(400000 - ((AV_{t-1} + 5160) \times 0.99 - CoI_t) * 1.05) \\
\left(1 - \frac{1.05}{1.06}q_t\right) CoI_t &= q_t(1.06)^{-1}(400000 - (AV_{t-1} + 5160) \times 0.99) \\
CoI_t &= \frac{q_t}{1.06 - 1.05q_t}(400000 - (AV_{t-1} + 5160) \times 0.99)
\end{aligned}$$

$$AV_t = 1.05\left(0.99(AV_{t-1} + 5160) - \frac{q_t}{1.06 - 1.05q_t}(400000 - (AV_{t-1} + 5160) * 0.99)\right)$$

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Profit signature is given by

| Year | Profit vector | Probability in force | Profit signature |
|---|---|---|---|
| 1 | 286.42 | 1.0000000 | 286.42 |
| 2 | 451.34 | 0.9489626 | 428.30 |
| 3 | 563.86 | 0.9004360 | 507.72 |
| 4 | 764.88 | 0.8812669 | 674.06 |
| 5 | 976.61 | 0.8623908 | 842.22 |
| 6 | 1193.12 | 0.8437907 | 1006.74 |
| 7 | 1420.48 | 0.8254506 | 1172.53 |
| 8 | 1660.71 | 0.7991134 | 1327.10 |
| 9 | 1907.88 | 0.7734466 | 1475.64 |
| 10 | 2161.98 | 0.7484175 | 1618.06 |
| 11 | 2423.01 | 0.7239944 | 1754.25 |
| 12 | 2690.95 | 0.7001450 | 1884.05 |

At the risk discount rate the NPV is

$$286.42(1.15)^{-1}+428.30(1.15)^{-2}+507.72(1.15)^{-3}+674.06(1.15)^{-4}+842.22(1.15)^{-5}+1006.74(1.15)^{-6}+1172.53(1.15)^{-7}+1327.10(1.15)$$

The EPV of the premiums received is

$$5180(1+0.9489626(1.15)^{-1}+0.9004360(1.15)^{-2}+0.8812669(1.15)^{-3}+0.8623908(1.15)^{-4}+0.8437907(1.15)^{-5}+0.8254506(1.15)^{-6}+0.7$$

The profit margin is $\frac{1769.39}{28233.07} = 0.063$

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After the premium is payed, the account value is \$123,389. The Expense charge is \$1,233.89, so after this charge, the account value is \$122,155.11. After applying the cost of insurance and interest, the account value is $\$130,705.97 - 1.07CoI$. The death benefit is the larger of $2.3(130,705.97 - 1.07CoI)$ and \$300,000. We first suppose that the death benefit is \$300,000, and calculate the cost of insurance charge. The Cost of insurance satisfies:

$$\begin{aligned}
CoI &= 0.000265(300000 - (130,705.97 - 1.07CoI))(1.06)^{-1} \\
&= 42.32 + 0.0002675CoI \\
0.9997325CoI &= 42.32 \\
CoI &= \frac{42.32}{0.9997325} = 42.33
\end{aligned}$$

With this cost of insurance, the account value at the end of the year is $130,705.97 - 1.07CoI = 130660.65$. This means that using the corridor factor requirement, the minimum death benefit is $2.3 \times 130660.65 = \$300,519.50$, so the corridor factor requirement means that the death benefit is increased. We need to calculate the Cost of Insurance based on the new death benefit.

$$\begin{aligned}
CoI &= 0.000265(1.3(130,705.97 - 1.07CoI))(1.06)^{-1} \\
&= 42.48 - 0.00034775CoI \\
1.00034775CoI &= 42.48 \\
CoI &= \frac{42.48}{1.00034775} = 42.46
\end{aligned}$$

## 14.3 Deterministic Profit Testing for Equity-Linked Insurance

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See slide.

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See slide.

## 14.4 Stochastic Profit Testing

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See slide.

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See slides.

## 14.5 Stochastic Pricing

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See slide.

## 14.6 Stochastic Reserving

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(a) 1246.704

(b) 3610.32

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See slide.