ACSC/STAT 4720, Life Contingencies II
Fall 2015
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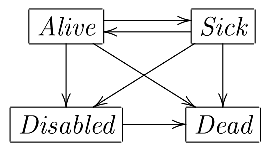
Homework Sheet 1
Model Solutions

Basic Questions

1. *An insurance company is developing a new policy. The policy considers 4 states: Healthy, Sick, Disabled, and Dead. The transition diagram is shown below:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) Alive—Disabled—Sick—Dead

(ii) Alive—Disabled—Dead

(iii) Alive—Sick—Alive—Disabled

(iv) Alive—Sick—Dead—Disabled

(i) is impossible since transitions from disabled to sick are not possible.

(ii) is possible.

(iii) is possible.

(iv) is impossible since transitions from dead to disabled are not possible.

2. *Consider a permanent disability model with transition intensities*

$$\mu_x^{01} = 0.001 + 0.000003x$$
$$\mu_x^{02} = 0.001 + 0.0000001x^2$$
$$\mu_x^{12} = 0.003 + 0.000005x$$

where State 0 is healthy, State 1 is permanently disabled and State 2 is dead. Calculate the probability that an individual aged 29 is alive but permanently disabled at age 56.

To be alive but permanently disabled at age 56, the individual must be healthy until age x, then become disabled at age x, then remain disabled until age 56. We integrate over all possible values of x:

$$
\begin{aligned}
{}_{27}p_{29}^{(01)} &= \int_{29}^{56} {}_{x-29}p_{29}^{\overline{00}}\mu_x^{01}\,{}_{56-x}p_x^{\overline{11}}\,dx \\
&= \int_{29}^{56} e^{-\int_{29}^{x} 0.002+0.000003y+0.0000001y^2\,dy}(0.001+0.000003x)e^{-\int_x^{56} 0.003+0.000005y\,dy}\,dx \\
&= \int_{29}^{56} e^{-(0.002(x-29)+0.0000015(x^2-29^2)+0.000000033333(x^3-29^3))}(0.001+0.000003x)e^{-(0.003(56-x)+0.0000025(56^2-x^2)}\,dx \\
&= \int_{29}^{56} (0.001+0.000003x)e^{0.001x+0.000001x^2-0.000000033333x^3-0.116013}\,dx \\
&= 0.02826281
\end{aligned}
$$

3. *Under a disability income model with transition intensities*

$$
\begin{aligned}
\mu_x^{01} &= 0.001 \\
\mu_x^{10} &= 0.002 \\
\mu_x^{02} &= 0.001 \\
\mu_x^{12} &= 0.003
\end{aligned}
$$

calculate the probability that a healthy individual dies within the next 7 years. [State 0 is healthy, State 1 is sick and State 2 is dead.]

We sum over the number of transitions between healthy and sick before dying. If there are n transitions from healthy to sick, and n transitions from sick to healthy, the probability density of any particular sequence of transitions is $0.001^n 0.002^n e^{-0.002h}e^{-0.005s}$, where h and s are the amounts of time spent in the healthy and sick states respectively.

Integrating over all possible sequences of transitions, the probability density of n transitions each way and total time h in the healthy state is

$$
\begin{aligned}
&0.001^n 0.002^n e^{-0.002h}e^{-0.005s}\int_{\substack{a_1+\ldots+a_n=h \\ b_1+\ldots+b_n=s}} da_1\cdots da_n\,db_1\cdots db_n \\
&= 0.001^n 0.002^n e^{-0.002h}e^{-0.005s}\left(\frac{s^{n-1}h^n}{(n!)^2}\right)
\end{aligned}
$$

The probability of no transitions is

$$
\int 0.001^n 0.002^n e^{-0.002h}e^{-0.005s}\left(\frac{s^{n-1}h^n}{(n!)^2}\right)
$$

The probability of being in the heathy state at time t is therefore

$$\sum_{n=0}^{\infty} 0.001^{n+1} 0.002^{n} \int_0^t e^{-0.002h} e^{-0.005(t-h)} \left(\frac{(t-h)^n h^n}{(n!)^2} \right) dh$$

The probability of being in the sick state at time t is

$$\sum_{n=0}^{\infty} 0.001^{n} 0.002^{n} \int_0^t e^{-0.002h} e^{-0.005(t-h)} \left(\frac{(t-h)^n h^{n+1}}{(n!)(n+1)!} \right) dh$$

The probability density of dying at time t is therefore

$$0.001 \sum_{n=0}^{\infty} 0.001^{n} 0.002^{n} \int_0^t e^{-0.002h} e^{-0.005(t-h)} \left(\frac{(t-h)^n h^{n+1}}{(n!)(n+1)!} \right) dh + 0.003 \sum_{n=0}^{\infty} 0.001^{n+1} 0.002^{n} \int_0^t e^{-0.002h} e^{-0.005(t-h)} \Bigg((t -$$

$$= \sum_{n=0}^{\infty} 0.001^{n} 0.002^{n} \int_0^t e^{-0.002h} e^{-0.005(t-h)} \left(\frac{(t-h)^n h^n}{(n!)^2} \right) \left(0.001 + 0.003 \times 0.001 \frac{h}{n+1} \right) dh$$

We see that this integral is the density of a sum of gamma distributions

We need to integrate this from 0 to 7. This gives

$$\int_0^7 \sum_{n=0}^{\infty} 0.001^{n} 0.002^{n} \int_0^t e^{-0.002h} e^{-0.005(t-h)} \left(\frac{(t-h)^n h^n}{(n!)^2} \right) \left(0.001 + 0.003 \times 0.001 \frac{h}{n+1} \right) dh\, dt$$

Rearranging the order of integration, and letting $s = t - h$, this becomes

$$\sum_{n=0}^{\infty} 0.001^{n} 0.002^{n} \int_0^7 \int_0^{7-h} e^{-0.002h} e^{-0.005s} \left(\frac{s^n h^n}{(n!)^2} \right) \left(0.001 + 0.003 \times 0.001 \frac{h}{n+1} \right) ds\, dh$$

$$= \sum_{n=0}^{\infty} 0.001^{n} 0.002^{n} \int_0^7 \left(\int_0^{7-h} e^{-0.005s} \left(\frac{s^{n-1}}{n!} \right) ds \right) e^{-0.002h} \frac{h^n}{n!} \left(0.001 + 0.003 \times 0.001 \frac{h}{n+1} \right) dh$$

Integrating this numerically using the following R-code:

```
ans<-0
for(n in 1:100){
sum(200^(n+1)*pgamma(((7-h)/200),(n+1))/exp(h/500)*h^n/gamma(n+1)*(0.001+0.000003*h/(n+1
ans<-ans+sum(200^(n+1)*pgamma(((7-h)/200),(n+1))/exp(h/500)*h^n/gamma(n+1)*(0.001+0.00000
}
```

gives 0.0000001131398. The $n = 0$ case gives

$$\begin{aligned}
&\int_0^7 0.001e^{-0.002h}\,dh + \int_0^7 \int_0^t 0.003 \times 0.001e^{-0.002h}e^{-0.005(t-h)}\,dh, dt \\
&= 0.5(1-e^{-0.014}) + \int_0^7 0.003 \times 0.001e^{-0.005t}\int_0^t e^{0.003h}\,dh\,dt \\
&= 0.5(1-e^{-0.014}) + \int_0^7 0.003 \times 0.001e^{-0.005t}\left(\frac{e^{0.003t}-1}{0.003}\right)dt \\
&= 0.5(1-e^{-0.014}) + \int_0^7 0.001(e^{-0.002t} - e^{-0.005t})\,dt \\
&= 0.5(1-e^{-0.014}) + 0.5(1-e^{-0.014}) - 0.2(1-e^{-0.035}) \\
&= 0.007023539
\end{aligned}$$

so

$$ {}_7p_x^{(02)} = 0.007023539 + 0.0000001131398 = 0.007024 $$

solution for people who have taken a course in stochastic processes:
The transition matrix is

$$\begin{pmatrix} -0.002 & 0.001 & 0.001 \\ 0.002 & -0.005 & 0.003 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0.2978 & -1.4244 & 1 \\ -1.0606 & -0.7999 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} -0.0055616 & 0 & 0 \\ 0 & -0.0014384 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0.4574 & -0.8144 & 0.3571 \\ -0.6064 & -0.1703 & 0.7767 \\ 0 & 0 & 1 \end{pmatrix}$$

The transition probabilities over a 7-year period are therefore given by

$$e^{7\begin{pmatrix} -0.002 & 0.001 & 0.001 \\ 0.002 & -0.005 & 0.003 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$= \begin{pmatrix} 0.2978 & -1.4244 & 1 \\ -1.0606 & -0.7999 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} e^{-0.0389312} & 0 & 0 \\ 0 & e^{-0.0100688} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0.4574 & -0.8144 & 0.3571 \\ -0.6064 & -0.1703 & 0.7767 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0.98615 & 0.00683 & 0.007024 \\ 0.01366 & 0.96565 & 0.020685 \\ 0 & 0 & 1 \end{pmatrix}$$

So the probability of dying within the 7 years is 0.007024.

4. *Under a disability income model with transition intensities*

$$\begin{aligned} \mu_x^{01} &= 0.001 \\ \mu_x^{10} &= 0.002 \\ \mu_x^{02} &= 0.001 \\ \mu_x^{12} &= 0.003 \end{aligned}$$

calculate the premium for a 5-year policy with premiums payable continuously while the life is in the healthy state, which pays benefits continuously at a rate of $130,000 per year while the life is in the sick state, sold to a life in the healthy state. [State 0 is healthy, State 1 is sick and State 2 is dead.]

[Hint: the probability that the life is healthy t years from the start of the policy is $0.1362e^{-0.0055616t} + 0.8638e^{-0.0014384t}$*, and the probability that the life is sick t years from the start of the policy is* $0.2426e^{-0.0014384t} - 0.2426e^{-0.0055616t}$*.]*

We have that

$$\begin{aligned} a_{34:\overline{5}|}^{00} &= \int_0^5 \left(0.1362e^{-0.0055616t} + 0.8638e^{-0.0014384t}\right) e^{-0.04t}\,dt \\ &= \int_0^5 0.1362e^{-0.0455616t} + 0.8638e^{-0.0414384t}\,dt \\ &= 0.1362\int_0^5 e^{-0.0455616t}\,dt + 0.8638\int_0^5 e^{-0.0414384t}\,dt \\ &= \frac{0.1362}{0.0455616}\left[-e^{-0.0455616t}\right]_0^5 + \frac{0.8638}{0.0455616}\left[-e^{-0.0414384t}\right]_0^5 \\ &= \frac{0.1362}{0.0455616}\left(1 - e^{-0.227808}\right) + \frac{0.8638}{0.0414384}\left(1 - e^{-0.207192}\right) \\ &= 4.509935 \end{aligned}$$

and

$$
\begin{aligned}
a^{01}_{34:\overline{5}|} &= \int_0^5 \left(0.2426e^{-0.0014384t} - 0.2426e^{-0.0055616t}\right) e^{-0.04t}\,dt \\
&= \frac{0.2426}{0.0414384}\left(1-e^{-0.207192}\right) - \frac{0.2426}{0.0455616}\left(1-e^{-0.227808}\right) \\
&= 0.01083035
\end{aligned}
$$

The net premium is therefore

$$
\frac{0.01083035 \times 130000}{4.509935} = \$312.19
$$

5. *A whole life insurance policy can end either through death or withdrawl. The transition intensities are*

$$
\begin{aligned}
\mu_x^{01} &= 0.001 + 0.000002x \\
\mu_x^{02} &= 0.002 + 0.000001x
\end{aligned}
$$

Calculate the probability that an individual aged 43 withdraws from the policy.

The probability that the individual withdraws from the policy is

$$
\begin{aligned}
\int_{43}^{\infty} e^{-\int_{43}^{x} 0.003+0.000003y\,dy}(0.001+0.000002x)\,dx &= \int_{43}^{\infty} e^{-(0.003(x-43)+0.0000015(x^2-43^2))}(0.001+0.000002x)\,dx \\
&= \int_{43}^{\infty} (0.001+0.000002x)e^{-(0.0000015x^2+0.003x-0.129-0.0027735))}\,dx \\
&= \int_{43}^{\infty} (0.001+0.000002x)e^{-0.0000015(x^2+2000x-87849)}\,dx \\
&= \int_{43}^{\infty} (0.001+0.000002x)e^{-0.0000015((x+1000)^2-1087849)}\,dx \\
&= e^{1.631773}\int_{43}^{\infty} (0.000002(x+1000)-0.001)e^{-\frac{0.000003(x+1000)^2}{2}}\,dx
\end{aligned}
$$

$$
\begin{aligned}
= e^{1.631773}\left(-0.000002\left[\frac{e^{-\frac{0.000003(x+1000)^2}{2}}}{0.000003}\right]_{43}^{\infty} \right. & \left. -0.001\sqrt{\frac{2\pi}{0.000003}}\left(1-\Phi\left(1043\times\sqrt{0.000003}\right)\right)\right) \\
&= \frac{2}{3}e^{1.631773-0.0000015\times 1043^2} - e^{1.631773}\sqrt{\frac{2\pi}{3}}\left(1-\Phi\left(0.003129\right)\right) \\
&= \frac{2}{3} - e^{1.631773}\sqrt{\frac{2\pi}{3}}\left(1-\Phi\left(1043\sqrt{0.000003}\right)\right) \\
&= 0.4045939
\end{aligned}
$$

Standard Questions

6. *An insurance company is developing a new model for transition intensities in a disability income model. Under these transition intensities it calculates*

$$\begin{array}{rcl} \overline{A}^{02}_{34} = 0.14 & \overline{A}^{02}_{44} = 0.19 & \overline{A}^{12}_{44} = 0.21 \\ \overline{a}^{00}_{34} = 22.07 & \overline{a}^{00}_{44} = 19.30 & \overline{a}^{10}_{44} = 0.11 \\ \overline{a}^{01}_{34} = 0.64 & \overline{a}^{01}_{44} = 0.43 & \overline{a}^{11}_{44} = 17.32 \\ {}_{10}p^{00}_{34} = 0.934 & {}_{10}p^{01}_{34} = 0.022 & \delta = 0.03 \end{array}$$

Calculate the premium for a 10-year policy with continuous premiums payable while in the healthy state, which pays a continuous benefit while in the sick state, at a rate of $80,000 per year, and pays a death benefit of $280,000 immediately upon death.

If the rate of premium is P, the EPV of total premiums received is

$$P\overline{a}00_{34:\overline{10|}} = P\left(\overline{a}^{00}_{34} -_{10} p^{00}_{34}e^{-10\delta}\overline{a}^{00}_{44} -_{10} p^{00}_{34}e^{-10\delta}\overline{a}^{10}_{44}\right) = (22.07-0.934e^{-0.3}\times 19.30-0.022e^{-0.3}\times 0.11)P = 8.71407P$$

The total EPV of benefits are

$$\begin{aligned} &80000\overline{a}^{01}_{34:\overline{10|}} + 280000\overline{A}^{02}_{34:\overline{10|}} \\ &= 80000\left(\overline{a}^{01}_{34} -_{10} p^{00}_{34}\overline{a}^{01}_{44} -_{10} p^{01}_{34}\overline{a}^{11}_{44}\right) + 280000\left(\overline{A}^{02}_{34} -_{10} p^{00}_{34}e^{-0.3}\overline{A}^{02}_{44} -_{10} p^{01}_{34}e^{-0.3}\overline{A}^{12}_{44}\right) \\ &= 80000(0.64 - 0.934e^{-0.3} \times 0.43 - 0.022e^{-0.3} \times 17.32) + 280000(0.14 - 0.934e^{-0.3} \times 0.19 - 0.022e^{-0.3} \times 0.21) \\ &= 4815.30 + 1431.31 \\ &= 6246.61 \end{aligned}$$

The premium is therefore $\frac{6246.61}{8.71407} = \$716.84$.