ACSC/STAT 4720, Life Contingencies II
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Homework Sheet 4
Model Solutions

Basic Questions

1. *A life aged 46 has mortality given in the table below. The yield rate is in another table below*

x	l_x	d_x
46	10000.00	1.01
47	9998.99	1.10
48	9997.89	1.21
49	9996.69	1.32
50	9995.37	1.45

Term (years)	Yield rate
1	0.018
2	0.021
3	0.028
4	0.031
5	0.033

Calculate the net annual premium for a 5-year endowment insurance with benefit \$150,000 sold to this life.

The EPV of benefits is $150000(0.000101(1.018)^{-1}+0.000110(1.021)^{-2}+0.000110(1.028)^{-3}+0.000132(1.031)^{-4}+0.999537(1.033)^{-5}) = 127529.23$.

If the annual premium is P then the EPV of premiums is $P(1+0.999899(1.018)^{-1}+0.999789(1.021)^{-2}+0.999669(1.028)^{-3}+0.999537(1.031)^{-4}) = 4.746127P$.

The premium is therefore $P = \frac{127529.23}{4.746127} = \$26,870.17$.

2. *An insurance company sells N one-year life insurance policies to lives aged 60. The death benefit is \$130,000, payable at the end of the year to lives which die during the year. The company uses $q_{60} = 0.00022$ and $i = 0.04$ to calculate the premium for the policy. This results in a net premium of $130000 \times 0.00022(1.04)^{-1} = \$27.5$.*

However, q_{60} is an estimated probability based on past data, and the true value is normally distributed with mean 0.00022 and standard deviation 0.00002. The interest rate cannot be fixed, and the actual interest rate obtained is normally distributed with mean 0.04 and standard deviation 0.005.

Calculate the expected aggregate profit of the policies, and the variance of this aggregate profit.

If the actual mortality is q, and the interest rate is i, then the number of payments follows a binomial distribution with parameters N and q. The expected aggregate profit is therefore

$$\mathbb{E}(P|q,i) = 27.5N(1+i) - 130000Nq$$

The overall expected aggragate profit is therefore

$$27.5N\mathbb{E}(1+i) - 130000N\mathbb{E}(q) = 27.5N\mathbb{E}(1.04) - 130000N(0.00022) = 0$$

The variance of the aggregate profit is given by the law of total variance:

$$\mathrm{Var}(P) = \mathrm{Var}(\mathbb{E}(P|q,i)) + \mathbb{E}(\mathrm{Var}(P|q,i))$$

The conditional variance is $130000^2 Nq(1-q)$. The expected conditional variance is therefore

$$130000^2 N(\mathbb{E}(q)-\mathbb{E}(q^2)) = 130000^2 N(\mathbb{E}(q)-(\mathbb{E}(q))^2-\mathrm{Var}(q)) = 130000^2 N(0.00022-0.00022^2-0.00002^2) = 3717175N$$

The variance of the conditional expectation (assuming q and i are independent) is

$$27.5^2 N^2 \operatorname{Var}(i) + 130000^2 N^2 \operatorname{Var}(q) = (27.5^2 \times 0.005^2 + 130000^2 \times 0.00002^2)N^2 = 6.778906N^2$$

The total variance of the aggregate profit is therefore $3717175N + 6.778906N^2$.

3. *An insurance company sells a pension to a life aged 66. The life has mortality following a Gompertz law with $B = 0.00000091$ and $C = 1.11$. The annual pension payment is currently \$65,000. Every year this payment increases by inflation, which follows a log-normal distribution with $\mu = 0$ and $\sigma^2 = 0.04$ (and inflation in each year is independent and independent of interest rates). The insurance company uses fixed income investments with a 10-year period to fix the interest rate, and the interest rates are reset every 10 years. The current interest rate is $i = 0.06$, and interest rates at future times are independent and normally distributed with mean 0.04 and standard deviation 0.01. Simulate 10000 PV future loss random variables.*

   (a) Use these to estimate the EPV future loss.

   If U follows a Uniform distribution, then we can model future lifetime by

$$\begin{aligned}
e^{-\frac{0.00000091}{\log(1.11)}1.11^{66}(1.11^T-1)} &= U \\
\frac{0.00000091}{\log(1.11)}1.11^{66}(1.11^T-1) &= -\log(U) \\
1.11^T - 1 &= -\log(U) \times \frac{\log(1.11)}{0.00000091(1.11)^{66}} \\
1.11^T &= 1 - \log(U) \times \frac{\log(1.11)}{0.00000091(1.11)^{66}} \\
T &= \frac{\log\left(1 - \log(U) \times \frac{\log(1.11)}{0.00000091(1.11)^{66}}\right)}{\log(1.11)}
\end{aligned}$$

   R code:

```
U<-runif(10000)
life<-log(1-log(U)*log(1.11)/(0.00000091*1.11^66))/log(1.11)
Klife<-ceiling(life)
paid<-rep(1,10000)%*%t(1:100)<Klife%*%t(rep(1,100))

loginf<-rnorm(1000000)*0.2
inf<-exp(loginf)
dim(inf)<-c(10000,100)
cuminf<-cbind(1,t(apply(inf,1,cumprod)))
```

```
irate<-rep(1,10)%*%t(rnorm(100000)*0.01+0.04)
dim(irate)<-c(100,10000)
irate<-t(irate)
irate<-cbind(rep(0.06,10000)%*%t(rep(1,10)),irate)
cumint<-cbind(1,t(apply(1+irate,1,cumprod)))

EPV<-65000*rowSums(paid*cuminf[,1:100]/cumint[,1:100])
```

More straightforward R-Code:

```
U<-runif(10000)
life<-log(1-log(U)*log(1.11)/(0.00000091*1.11^66))/log(1.11)
Klife<-floor(life)
#Klife is number of whole years survived.

EPV<-rep(0,10000)

for(i in 1:10000){
  #for each simulation
   loginf<-rnorm(Klife(i))*0.2
   inf<-exp(loginf)
   cuminf<-cbind(1,cumprod(inf))

  #cumulative product of annual inflations

   numint<-floor(Klife[i]/10)
   irate<-as.vector(rep(1,10)%*%t(c(0.06,rnorm(numint)*0.01+0.04)))
  #repeat each interest rate 10 times.
   cumint<-cbind(1,cumprod(irate)[1:Klife[i]])
  #cumulative product of interest rates

  EPV[i]<-65000*sum(cuminf/cumint)
}
```

Taking the average of the PV future loss in the simulations, in my simulation gives \$1,548,558.21.

(b) Construct a 95% confidence interval for the EPV future loss.

For my simulation, the variance of the EPV is 1,921,513,198,729. so the standard deviation is 1386187. Since we sampled 10000 simulations, the standard deviation of the mean of this is 13861.87. By the central limit theorem, the distribution is approximately normal, so a 95% confidence interval is the mean plus or minus 1.96 standard deviations, that is

$$1548558.21 \pm 1.96 \times 13861.87 = [\$1,521,389.46, \$1,575,726.97]$$

Standard Questions

4. *An insurance company sells a large number of 4-year life insurance policies to lives aged 54, with the following lifetable*

x	l_x	d_x
54	10000.00	4.95
55	9995.05	5.34
56	9989.71	5.76
57	9983.95	6.23
58	9977.71	6.75

They can fix the interest rate for the first two years at 4%. For the last two years, they can either fix the interest rate for the last two years at 4%, or they can use the market rates in two years' time. If they do this, the market rates will follow a log-normal distribution with $\mu = -3.8$ *and* $\sigma^2 = 0.3$. *What is the probability that it does worse by using market rates than by fixing the rates at* $i = 0.04$*?*

The insurance company does worse if the market rates are lower than 0.04. This happens if and only if the log market rates are lower than $\log(0.04)$. The probability of this is $\Phi\left(\frac{\log(0.04)+3.8}{\sqrt{0.3}}\right) = \Phi(1.060983) = 0.8556512$.

5. *An insurance company has* N *10-year term insurance policies sold to lives aged 37 with mortality following the lifetable below.*

x	l_x	d_x
37	10000.00	2.39
38	9997.61	2.41
39	9995.19	2.44
40	9992.76	2.46
41	9990.30	2.48
42	9987.82	2.51
43	9985.30	2.54
44	9982.76	2.57
45	9980.19	2.61
46	9977.58	2.65

The current interest rate is $i = 0.04$ *and is fixed for 5 years. After 5 years, the insurance company will fix interest rates at the new interest rate for the remainder of the policy. The new interest rate is normally distributed with mean 0.03 and variance 0.005. For what value of* N *is the variance due to uncertainty in the interest rate larger than the variance due to uncertainty in mortality?*

Suppose that the premium is P and the death benefit is 1. The EPV of the policy is

$$0.000239(1.04)^{-1}+0.000241(1.04)^{-2}+0.000244(1.04)^{-3}+0.000246(1.04)^{-4}+0.000248(1.04)^{-5}+0.000251(1.04)^{-5}(1+i)^{-1}+0.00025$$

In theory, we need to calculate the expected value of this to calculate the premium. Since i is small, it is a good approximation to use the expected value of i. This gives

$$0.002052537 - 8.498222P$$

so $P = \frac{0.002052537}{8.498222} = 0.0002415255$

Mortality follows a binomial distribution, which can be approximated by a Poisson distribution, or a normal distribution with variance equal to the mean. Let the observed number of deaths per policy sold in year n be D_n. Now D_n is approximately normally distributed with mean q_{36+n} (we number years starting at 1) and variance $\frac{q_{36+n}}{N}$. Since overall mortality is small, we can assume the D_n are independent. The total profit is therefore (letting $S_n = \sum_{j=1}^{n} D_j$)

$$P\left(1+\frac{1-S_1}{1.04}+\frac{1-S_2}{1.04^2}+\frac{1-S_3}{1.04^3}+\frac{1-S_4}{1.04^4}+\frac{1-S_5}{1.04^5}+\frac{1-S_6}{1.04^5(1+i)}+\frac{1-S_7}{1.04^5(1+i)^2}+\frac{1-S_8}{1.04^5(1+i)^3}+\frac{1-S_9}{1.04^5(1+i)^4}\right)$$
$$-\left(\frac{D_1}{1.04}+\frac{D_2}{1.04^2}+\frac{D_3}{1.04^3}+\frac{D_4}{1.04^4}+\frac{D_5}{1.04^5}+\frac{D_5}{1.04^5(1+i)}+\frac{D_6}{1.04^5(1+i)^2}+\frac{D_8}{1.04^5(1+i)^3}+\frac{D_9}{1.04^5(1+i)^4}+\frac{D_{10}}{1.04^5(1+i)^5}\right)$$

For $i = 0.03$, this becomes:

$$\begin{aligned}
&8.507006P\\
&-\left(\frac{D_1}{1.04}+\frac{D_2}{1.04^2}+\frac{D_3}{1.04^3}+\frac{D_4}{1.04^4}+\frac{D_5}{1.04^5}+\frac{D_5}{1.04^5 1.03}+\frac{D_6}{1.04^5 1.03^2}+\frac{D_8}{1.04^5 1.03^3}+\frac{D_9}{1.04^5 1.03^4}+\frac{D_{10}}{1.04^5 1.03^5}\right)\\
&-\left(\frac{S_1}{1.04}+\frac{S_2}{1.04^2}+\frac{S_3}{1.04^3}+\frac{S_4}{1.04^4}+\frac{S_5}{1.04^5}+\frac{S_5}{1.04^5 1.03}+\frac{S_6}{1.04^5 1.03^2}+\frac{S_8}{1.04^5 1.03^3}+\frac{S_9}{1.04^5 1.03^4}+\frac{S_{10}}{1.04^5 1.03^5}\right)P\\
&= 8.507006P - D_1\left(\frac{1}{1.04}+7.507006P\right)-\cdots-D_9\left(\frac{1}{1.04^5 1.03^4}+\frac{1}{1.04^5 1.03^4}P\right)-D_{10}\left(\frac{1}{1.04^5 1.03^5}\right)
\end{aligned}$$

The variance of this is therefore

$$\begin{aligned}
&\left(\frac{1}{1.04}+7.507006P\right)^2 \mathrm{Var}(D_1)+\cdots+\left(\frac{1}{1.04^5 1.03^5}+\frac{1}{1.04^5 1.03^5}P\right)^2 \mathrm{Var}(D_{10})\\
&= \frac{1}{N}\left(0.9633516^2 \times 0.000239 + 0.9261371^2 \times 0.000241 + 0.8903540^2 \times 0.000244 + 0.8559471^2 \times 0.000246 + 0.8228635^2 \times 0.0002\right.\\
&= \frac{0.001695498}{N}
\end{aligned}$$

If we fix the mortality as in the table, then the EPV of the profit at interest rate i is

$$0.000239(1.04)^{-1}+0.000241(1.04)^{-2}+0.000244(1.04)^{-3}+0.000246(1.04)^{-4}+0.000248(1.04)^{-5}+0.000251(1.04)^{-5}(1+i)^{-1}+0.00025$$

This is

$0.001083661 + 5.448674P + (0.0002063037 + 0.8207189P)(1+i)^{-1} + (0.0002087695 + 0.8205101P)(1+i)^{-2} + (0.0002112353 + 0.8202989P)(1+i)^{-3} + (0.0002145230 + 0.8200843P)(1+i)^{-4} + 0.0002178107(1+i)^{-5}$

Substituting $P = 0.0002415255$ gives:

$$0.002399655+0.0004045282(1+i)^{-1}+0.0004069436(1+i)^{-2}+0.0004093584(1+i)^{-3}+0.0004125943(1+i)^{-4}+0.0002178107(1+i)^{-5}$$

i is normally distibuted with mean 0.03 and standard deviation 0.005. We therefore calculate the variance value of this by numerically integrating

$$\frac{1}{0.005\sqrt{2\pi}}\int_0^\infty (0.0004045282((1+i)^{-1}-(1.03)^{-1})+0.0004069436((1+i)^{-2}-(1.03)^{-2})+0.0004093584((1+i)^{-3}-(1.03)^{-3})+0.00041$$

The variance for N policies due to interest rate uncertainty is therefore $1.638954\times10^{-11}N^2$. We want to know for what value of N is this larger than $0.001695498N$. This happens when $N>\frac{0.001695498}{1.638954\times10^{-11}}=103450005$