8.5 Numerical Evaluation of Probabilities

1

Density of event individual became disabled at time t is

$$_tp_{27}^{00}\mu_{27+t}^{01}\,_{16-t}p_{27+t}^{11}$$

so probability is

$$\int_0^{16} 0.003e^{-0.004t}e^{-0.002(16-t)}dt = 0.003e^{-0.032}\int_0^{16}e^{-0.002t}dt = \frac{0.003}{0.002}e^{-0.032}(1-e^{-0.032}) = 0.04575237$$

2

Density of event individual became disabled at time t is

$$_tp_{32}^{00}\mu_{32+t}^{01}\,_{12-t}p_{32+t}^{11}$$

We have

$$_tp_x^{00} = e^{-\int_0^t 0.004+0.00003(x+s)ds} = e^{-(0.004+0.00003x)t+0.000015t^2}$$

$$_tp_x^{11} = e^{-\int_0^t 0.002+0.00002(x+s)ds} = e^{-(0.002+0.00002x)t+0.00001t^2}$$

so probability is

$$\int_0^{12} (0.001+0.00001(32+t))e^{-(0.00496t+0.000015t^2)}e^{-(0.00264(12-t)+0.00001(12-t)^2)}dt$$

$$= \int_0^{12} (0.001+0.00001(32+t))e^{-(0.00496t+0.000015t^2)}e^{-((0.00264+0.00002t)(12-t)+0.00001(12-t)^2)}dt$$

$$= \int_0^{12} (0.001+0.00001(32+t))e^{-(0.03312+0.0028t+0.000005t^2)}dt$$

$$= \int_0^{12} (0.001+0.00001(32+t))e^{-\frac{((t+28)^2)-121.6}{20000}}dt$$

$$= \frac{e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}(132+t)e^{-\frac{((t+28)^2)}{20000}}dt$$

$$= \frac{e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}(104+(t+28))e^{-\frac{((t+28)^2)}{20000}}dt$$

$$= \frac{104e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}e^{-\frac{((t+28)^2)}{20000}}dt + \frac{e^{-\frac{121.6}{20000}}}{10000}\left[-10000e^{-\frac{((t+28)^2)}{20000}}\right]_0^{12}$$

$$= \frac{104}{100}e^{-\frac{121.6}{20000}}\sqrt{2\pi}\left(\Phi\left(\frac{28}{100}\right)-\Phi\left(\frac{16}{100}\right)\right) + e^{-\frac{121.6}{20000}}\left(e^{-\frac{28^2}{20000}}-e^{-\frac{40^2}{20000}}\right)$$

$$= 0.1250398$$

3

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We have

$$_tp_x^{\overline{00}} = e^{-0.0004t}$$

$$_tp_x^{\overline{11}} = e^{-0.00023t}$$

Probability this happens from 1 transition:

$$\int_0^{16} {}_tp_{27}^{\overline{00}}\mu_{27+t}^{01}{}_{(16-t)}p_{27+t}^{\overline{11}}\, dt$$

$$= \int_0^{16} 0.0003e^{-0.0004t}e^{-0.00023(16-t)}\, dt$$

$$= 0.0003e^{-0.00368}\int_0^{16} e^{-0.00017t}\, dt$$

$$= \frac{0.0003}{0.00017}e^{-0.00368}\left(1 - e^{-0.00272}\right)$$

Probability this happens from 3 transitions:

$$\int_0^{16}\int_s^{16}\int_t^{16} {}_sp_{27}^{\overline{00}}\mu_{27+s}^{01}{}_{(t-s)}p_{27+s}^{\overline{11}}\mu_{27+t}^{10}{}_{u-t}p_{27+t}^{\overline{00}}\mu_{27+u}^{01}{}_{(16-u)}p_{27+u}^{\overline{11}}\, du\, dt\, ds$$

$$= \int_0^{16}\int_s^{16}\int_t^{16} 0.0003^2 \times 0.00003e^{-0.0004(s+u-t)}e^{-0.00023(16-u+t-s)}\, du\, dt\, ds$$

$$= 0.0003^2 \times 0.00003e^{-0.00368}\int_0^{16}\int_s^{16}\int_t^{16} e^{-0.00017(s+u-t)}\, du\, dt\, ds$$

Making the substitution $w = s + u - t$, this becomes (noting that $|J| = 1$):

$$0.0003^2 \times 0.00003e^{-0.00368}\int_0^{16}\int_0^{w}\int_s^{16+s-w} e^{-0.00017w}\, dt\, ds\, dw$$

$$0.0003^2 \times 0.00003e^{-0.00368}\int_0^{16} e^{-0.00017w}\int_0^{w}\int_s^{16+s-w} 1\, dt\, ds\, dw$$

$$0.0003^2 \times 0.00003e^{-0.00368}\int_0^{16} w(16-w)e^{-0.00017w}\, dw$$

In general, the probability that this happens from $2n + 1$ transitions is

$$0.0003\,(0.00003 \times 0.0003)^n\, e^{-0.00368}\int_0^{16}\frac{w^n(16-w)^n}{(n!)^2}e^{-0.00017w}\, dw$$

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The total probability of this is therefore

$$\sum_{n=0}^{\infty} 0.0003 \left(0.00003 \times 0.0003\right)^n e^{-0.00368} \int_0^{16} \frac{w^n (16-w)^n}{(n!)^2} e^{-0.00017w} \, dw$$

$$= 0.004775872$$

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We have that $_4p_{37}^{(01)} = {_2p_{37}^{(00)}} \times {_2p_{39}^{(01)}} + {_2p_{37}^{(01)}} \times {_2p_{39}^{(11)}}$. For the numbers in the table, this gives

$$0.007857 = 0.992036 \times 0.003960 + 0.003960 \times 0.992054 = 0.007857$$
$$0.007857 = 0.992036 \times 0.003968 + 0.003968 \times 0.992054 = 0.007873$$
$$0.007857 = 0.992036 \times 0.003964 + 0.003964 \times 0.990054 = 0.007857$$

So the second actuary's calculations cannot be right. Furthermore, since $\mu_x^{02} < \mu_x^{12}$ for all x, we should have $_4p_{37}^{02} <_4 p_{37}^{12}$, which rules out the first actuary's calculations. This means that only Actuary III's calculations might be correct. [Indeed these are the correct values.]

8.6 Premiums

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The rate of exit of state 0 is $0.0004 + 0.000003(t+42) = 0.000526 + 0.000003t$, so $_tp_{42}^{(00)} = e^{-\int_0^t 0.000526 + 0.000003t \, dt} = e^{-0.000526 - 0.0000015t^2}$.

Premiums are payable while healthy. If the rate of premium is P, then the expected present value of the premium paid is

$$P \int_0^5 e^{-0.03t} e^{-0.000526t - 0.0000015t^2} \, dt$$

$$= P \int_0^5 e^{-0.030526t - 0.0000015t^2} \, dt$$

$$= P \int_0^5 e^{-0.0000015(t + 10175.333333)^2 + 0.0000015 \times 10175.333333^2} \, dt$$

$$= P e^{0.0000015 \times 10175.333333^2} \sqrt{\frac{\pi}{0.0000015}} \left(\Phi \left(10180.333333\sqrt{0.000003}\right) - \Phi \left(10175.333333\sqrt{0.000003}\right) \right)$$

$$= 4.637064P$$

On the other hand, the expected death benefits for lives that are not critically ill first are:

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$$100000 \int_0^5 e^{-0.03t} e^{-0.000526t - 0.0000015t^2} (0.0001 + 0.000001(t+42)) \, dt$$

$$= 100000 \int_0^5 e^{-0.030526t - 0.0000015t^2} (0.000142 + 0.000001t) \, dt$$

$$= 100000 \int_0^5 e^{-0.030526t - 0.0000015t^2} \left(0.000001\left(t + \frac{0.030526}{0.000003}\right) + 0.000142 - 0.01017533333333\right) dt$$

$$= 0.1 e^{\frac{0.030526^2}{0.000006}} \int_0^5 \left(t + \frac{0.030526}{0.000003}\right) e^{-0.0000015\left(t + \frac{0.030526}{0.000003}\right)^2} dt - 1003.333333333333 \int_0^5 e^{-0.030526t - 0.0000015t^2} \, dt$$

$$= 0.1 e^{\frac{0.030526^2}{0.000006}} \left[-\frac{e^{-0.0000015\left(t + \frac{0.030526}{0.000003}\right)^2}}{0.000003} \right]_0^5 - 1003.333333333333 \times 4.637064$$

$$= 66.97587$$

If a life becomes critically ill at age x, the probability that it survives for t years is $e^{-\int_0^t 0.02 \, dt} = e^{-0.02t}$. The expected value of the benefits to such a life is therefore given by

$$90000 \int_0^{47-x} e^{-0.03(x+t-42)} e^{-0.02t} \, dt = 90000 e^{-0.03(x-42)} \int_0^{47-x} e^{-0.05t} \, dt = 1800000 e^{-0.03(x-42)} (1 - e^{-0.05(47-x)})$$

The expected value of death benefits to such an individual is

$$100000 \int_0^{47-x} 0.02 e^{-0.03(x+t-42)} e^{-0.02t} \, dt = 2000 e^{-0.03(x-42)} \int_0^{47-x} e^{-0.05t} \, dt = 4000 e^{-0.03(x-42)} (1 - e^{-0.05(47-x)})$$

So the total expected benefits paid to individuals who become disabled are

$$1840000 \int_0^5 e^{-0.000526s - 0.0000015s^2} (0.0003 + 0.000002(s+42)) e^{-0.03(5-s)} (1 - e^{-0.05(5-s)}) \, ds$$

$$= 1.84 \int_0^5 (384 + 2s) e^{-0.15 + 0.029474s - 0.0000015s^2} (1 - e^{-0.05(5-s)}) \, ds$$

$$= 1.84 \int_0^5 (384 + 2s) \left(e^{-0.15 + 0.029474s - 0.0000015s^2} - e^{-0.4 + 0.079474s - 0.0000015s^2} \right) ds$$

$$0.15 - 0.029474s + 0.0000015s^2 = 0.0000015(s^2 - 19649.333333s + 100000)$$
$$= 0.0000015((s - 9824.666667)^2 - 96424075)$$
$$0.4 - 0.079474s + 0.0000015s^2 = 0.0000015(s^2 - 52982.666667s + 266666.666667)$$
$$= 0.0000015((s - 26491.333333)^2 - 701524075)$$

So the expected benefits are

$$1.84 \int_0^5 (384 + 2s)(e^{144.6361}e^{-\frac{0.000003(s-9824.666667)^2}{2}} - e^{1052.286}e^{-\frac{0.000003(s-26491.333333)^2}{2}})\, ds$$

$$= 1.84e^{144.6361} \int_0^5 (20033.3333333 + 2(s - 9824.67))e^{-\frac{0.000003(s-9824.67)^2}{2}}\, ds - 1.84e^{1052.286} \int_0^5 (53366.67 + 2(s - 26491.33))e^{-\frac{0.000003(s}{}}$$

$$= 1.84e^{144.6361} \left(20033.33 \times \sqrt{\frac{2\pi}{0.000003}} \left(\Phi(-9819.67 \times \sqrt{0.000003}) - \Phi(-9824.67 \times \sqrt{0.000003}) \right) - 2 \left[\frac{e^{-\frac{0.000003(s-9824.67)^2}{2}}}{0.000003} \right]_0^5 \right)$$

$$- 1.84e^{1052.286} \left(53366.67 \times \sqrt{\frac{2\pi}{0.000003}} \left(\Phi(-26491.33 \times \sqrt{0.000003}) - \Phi(-26496.33 \times \sqrt{0.000003}) \right) - 2 \left[\frac{e^{-\frac{0.000003(s-26491.33)^2}{2}}}{0.000003} \right]_0^5 \right)$$

$$= 371.8703$$

The total expected benefit is therefore

$$371.8703 + 66.97587 = 438.8461$$

The premium is therefore

$$\frac{438.8461}{4.637572} = \$94.64$$

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$$0.000001 \begin{pmatrix} -\left(1843x + 38x^2 + \frac{x^3}{3}\right) & 374x + x^2 & 1469x + 37x^2 + \frac{x^3}{3} \\ 67x + 0.5x^2 & -(341x + 1.5x^2) & 274x + x^2 \\ 0 & 0 & 0 \end{pmatrix}$$

We calculate the probability that the life is in each state at the end of each year:

t	$_tp_{37}^{00}$	$_tp_{37}^{01}$	$_tp_{37}^{02}$
0	1	0	0
1	0.99812	0.000375	0.001505
2	0.99617	0.000750	0.003083
3	0.99414	0.001127	0.004736
4	0.99203	0.001505	0.006464
5	0.98985	0.001884	0.008271
6	0.98758	0.002263	0.010156
7	0.98523	0.002644	0.012123
8	0.98280	0.003025	0.014171
9	0.98029	0.003407	0.016303
10	0.97769	0.003790	0.018519

EPV of death benefits is

$$200000\big(0.00150(1.06)^{-1} + 0.00157(1.06)^{-2} + 0.00165(1.06)^{-3} + 0.00172(1.06)^{-4} + 0.00180(1.06)^{-5} + 0.00188(1.06)^{-6}$$
$$+0.00197(1.06)^{-7} + 0.00205(1.06)^{-8} + 0.00213(1.06)^{-9} + 0.00222(1.06)^{-10}\big) = 2670.49$$

EPV of disability benefits is

$$80000\big(0.000375(1.06)^{-1} + 0.000750(1.06)^{-2} + 0.001127(1.06)^{-3} + 0.001505(1.06)^{-4} + 0.001884(1.06)^{-5} + 0.002263(1.06)^{-6}$$
$$+0.002644(1.06)^{-7} + 0.003025(1.06)^{-8} + 0.003407(1.06)^{-9} + 0.003790(1.06)^{-10}\big) = 1116.21$$

EPV of all benefits is $2670.49 + 1116.21 = 3786.70$

EPV of unit premiums is $1 + 0.99812(1.06)^{-1} + 0.99617(1.06)^{-2} + 0.99414(1.06)^{-3} + 0.99203(1.06)^{-4} +$
$0.98985(1.06)^{-5} + 0.98758(1.06)^{-6} + 0.98523(1.06)^{-7} + 0.98280(1.06)^{-8} + 0.98029(1.06)^{-9} = 7.736653311$

So annual premium is

$$\frac{3786.70}{7.736653311} = \$489.45$$

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If the rate of premium is P, the EPV of total premiums received is

$$P\bar{a}00_{34:\overline{10}|} = P\left(\bar{a}_{34}^{00} - {}_{10}p_{34}^{00}e^{-10\delta}\bar{a}_{44}^{00} - {}_{10}p_{34}^{00}e^{-10\delta}\bar{a}_{44}^{10}\right) = (22.07 - 0.934e^{-0.3}\times19.30 - 0.022e^{-0.3}\times0.11)P = 8.71407P$$

The total EPV of benefits are

$$80000\bar{a}_{34:\overline{10}|}^{01} + 280000\overline{A}_{34:\overline{10}|}^{02}$$
$$= 80000\left(\bar{a}_{34}^{01} - {}_{10}p_{34}^{00}\bar{a}_{44}^{01} - {}_{10}p_{34}^{01}\bar{a}_{44}^{11}\right) + 280000\left(\overline{A}_{34}^{02} - {}_{10}p_{34}^{00}e^{-0.3}\overline{A}_{44}^{02} - {}_{10}p_{34}^{01}e^{-0.3}\overline{A}_{44}^{12}\right)$$
$$= 80000(0.64 - 0.934e^{-0.3}\times0.43 - 0.022e^{-0.3}\times17.32) + 280000(0.14 - 0.934e^{-0.3}\times0.19 - 0.022e^{-0.3}\times0.21)$$
$$= 4815.30 + 1431.31$$
$$= 6246.61$$

The premium is therefore $\frac{6246.61}{8.71407} = \$716.84$.

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We calculate

$$\bar{a}_{x:\overline{5}|} = \int_0^5 e^{-0.03t}(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})\,dt$$
$$= \int_0^5 0.2113249e^{-0.036732051t}\,dt + \int_0^5 0.7886751e^{-0.033267949t})\,dt$$
$$= \frac{0.2113249}{0.036732051}(1 - e^{-0.036732051\times5}) + \frac{0.7886751}{0.033267949}(1 - e^{-0.033267949\times5})$$
$$= 4.598130$$

The EPV of the benefits to lives who die accidentally from State 0 are given by

$$200000 \int_0^5 0.002(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t} \, dt$$

$$= 400 \int_0^5 (0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t} \, dt$$

$$= 400 \times 4.598130$$

$$= 1839.25$$

The EPV of the benefits to lives who die otherwise from State 0 are given by

$$100000 \int_0^5 0.001(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t} \, dt$$

$$= 100 \int_0^5 (0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t} \, dt$$

$$= 100 \times 4.598130$$

$$= 459.81$$

The EPV of the benefits to lives who die accidentally from State 1 are given by

$$200000 \int_0^5 0.001(0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t})e^{-0.03t} \, dt$$

$$= 200 \int_0^5 0.2886752e^{-0.033267949t} - 0.2886752e^{-0.036732051t}) \, dt$$

$$= \frac{57.73504}{0.033267949}(1 - e^{-5 \times 0.033267949}) - \frac{57.73504}{0.036732051}(1 - e^{-5 \times 0.036732051})$$

$$= 2.226627$$

The EPV of the benefits to lives who die otherwise from State 1 are given by

$$100000 \int_0^5 0.003(0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t})e^{-0.03t} \, dt$$

$$= 300 \int_0^5 0.2886752e^{-0.033267949t} - 0.2886752e^{-0.036732051t}) \, dt$$

$$= 3.339940$$

The total EPV of benefits is therefore $1839.25 + 459.81 + 2.23 + 3.34 = \$2,304.63$.
The annual rate of premium is $\frac{2304.63}{4.598130} = \$501.21$.

8.7 Policy Values and Thiele's Differential Equation

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Thiele's differential equation gives

$$\frac{d}{dt}{}_tv^{(1)} = \delta_t v^{(1)} + P^{(1)} - B^{(1)} - \sum_{j \neq 1} \mu^{(1j)}_{x+t}(S^{(1j)} + {}_t v^{(j)} - {}_t v^{(1)})$$

$$= 0.03{}_tv^{(1)} - 90000 - \mu^{(12)}_{x+t}(100000 - {}_tv^{(1)})$$

$$= 0.03{}_tv^{(1)} - 90000 - 0.02(100000 - {}_tv^{(1)})$$

$$= 0.05{}_tv^{(1)} - 92000$$

$${}_tv^{(1)} = 1840000(1 - e^{0.05(t-5)})$$

$$\frac{d}{dt}{}_tv^{(0)} = \delta_t v^{(0)} + P^{(0)} - B^{(0)} - \sum_{j \neq 0} \mu^{(0j)}_{x+t}(S^{(0j)} + {}_t v^{(j)} - {}_t v^{(0)})$$

$$= 0.03{}_tv^{(0)} + 71.83 - 0 - \mu^{(01)}_{x+t}({}_tv^{(1)} - {}_t v^{(0)}) - \mu^{(02)}_{x+t}(100000 - {}_tv^{(0)})$$

$$= 0.03{}_tv^{(0)} + 71.83 - (0.0003 + 0.000002(42+t))({}_tv^{(1)} - {}_t v^{(0)}) - (0.0001 + 0.000001(42+t))(100000 - {}_tv^{(0)})$$

$$= 0.03{}_tv^{(0)} + 71.83 - (0.000384 + 0.000002t)(1840000(1 - e^{0.05(t-5)}) - {}_t v^{(0)}) - (0.000142 + 0.000001t)(100000 - {}_tv^{(0)})$$

$$= 71.83 - 706.56 - 14.2 - 3.68t(1 - e^{0.05(t-5)}) - 0.1t + (0.03 + 0.0000384 + 0.000142){}_tv^{(0)} + 0.000003t{}_tv^{(0)} + 70.656e^{0.05(t-5)}$$

$$= -13.026 - 3.78t + 3.68te^{0.05(t-5)} + 0.0300526{}_tv^{(0)} + 0.000003t{}_tv^{(0)} + 706.56e^{0.05(t-5)}$$

8.9 Multiple Decrement Tables

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We have ${}_tp_x^{(00)} = e^{-0.0023t-0.000005((x+t)^2-x^2)}$, so we have

$$\bar{a}^{(00)}_{36:\overline{10}|} = \int_0^{10} e^{-0.04t}e^{-0.0023t-0.000005((36+t)^2-36^2)}\,dt$$

$$= \int_0^{10} e^{-0.04266t-0.000005t^2}\,dt$$

$$= e^{\frac{4266}{200000}}\int_0^{10} e^{-\frac{(t+4266)^2}{200000}}\,dt$$

$$= 100\sqrt{20\pi}e^{\frac{4266^2}{20000}}\left(\Phi\left(\frac{42.76}{\sqrt{10}}\right) - \Phi\left(\frac{42.66}{\sqrt{10}}\right)\right) = 8.139325$$

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$$\overline{A}^{(01)}_{36:\overline{10}|} = \int_0^{10} (0.00102 + 0.00002t)e^{-0.04t}e^{-0.0023t - 0.000005((36+t)^2 - 36^2)}\, dt$$

$$= e^{\frac{4266^2}{200000}} \int_0^{10} 0.00002(t+51)e^{-\frac{(t+4266)^2}{200000}}\, dt$$

$$= 0.00002 e^{\frac{4266^2}{200000}} \left(\int_0^{10} (t+4266)e^{-\frac{(t+4266)^2}{200000}}\, dt - 4215 \int_0^{10} e^{-\frac{(t+4266)^2}{200000}}\, dt \right)$$

$$= 0.00002 e^{\frac{4266^2}{200000}} \left(\left[-100000 e^{-\frac{(t+4266)^2}{200000}} \right]_0^{10} - 421500\sqrt{20\pi} \left(\Phi\left(\frac{42.76}{\sqrt{10}}\right) - \Phi\left(\frac{42.66}{\sqrt{10}}\right) \right) \right)$$

$$= 0.009058283$$

So the annual rate of premium is

$$\frac{300000 \times 0.01093749}{8.007432} = \$333.87$$

(b) For policies which do not lapse, we only need to consider the mortality rate. This gives

$$\overline{a}^{(00)}_{36+t:\overline{10-t}|} = \int_0^{10-t} e^{-0.04s}e^{-0.0003s - 0.000005((36+t+s)^2 - (36+t)^2)}\, ds$$

$$= \int_0^{10-t} e^{-(0.04102+0.00001t)s - 0.000005s^2}\, ds$$

$$= e^{\frac{4102^2}{200000}} \int_0^{10-t} e^{-\frac{(s+t+4102)^2}{200000}}\, ds$$

$$= 100\sqrt{20\pi}\, e^{\frac{(t+4102)^2}{200000}} \left(\Phi\left(\frac{41.12}{\sqrt{10}}\right) - \Phi\left(\frac{t+4102}{100\sqrt{10}}\right) \right)$$

and

$$\overline{A}^{(00)}_{36+t:\overline{10-t}|} = 1 - 0.04\overline{a}^{(00)}_{36+t:\overline{10-t}|}$$

So the policy value is

$$_tV = 300000(1 - 0.04\overline{a}^{(00)}_{36+t:\overline{10-t}|} - _{10-t}p_{36+t}e^{-0.04(10-t)}) - P\overline{a}^{(00)}_{36+t:\overline{10-t}|}$$

$$= 300000 - (12000 + P)\overline{a}^{(00)}_{36+t:\overline{10-t}|} - 300000\,_{10-t}p_{36+t}e^{-0.04(10-t)}$$

$$= 300000 - (12000 + P)100\sqrt{20\pi}\, e^{\frac{(t+4102)^2}{200000}} \left(\Phi\left(\frac{41.12}{\sqrt{10}}\right) - \Phi\left(\frac{t+4102}{100\sqrt{10}}\right) \right) - 300000\,_{10-t}p_{36+t}e^{-0.04(10-t)}$$

$$= a(t) + b(t)P$$

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for some functions $a(t)$ and $b(t)$.

From (a), the expected death benefit of the policy is $300000 \times 0.01093749 = \$3281.25$.

The expected surrender benefit for the policy is

$$\frac{1}{2}\int_0^{10} e^{-0.04t} {}_tp_{36}\mu_{36+t}^{(01)} {}_tV\, dt = \frac{1}{2}\int_0^{10} e^{-0.04t} e^{-(0.0023t+0.000005(t^2+72t))}(0.002 - 0.00001(36+t)){}_tV\, dt$$

$$= \frac{1}{2}\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t)){}_tV\, dt$$

$$= \frac{1}{2}\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))(a(t)+b(t)P)\, dt$$

$$= \frac{1}{2}\left(\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))a(t)\, dt\right.$$

$$\left. + \int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))b(t)P\, dt\right)$$

$$= \frac{1}{2}(46.51242 - 0.08175562P)$$

Using numerical integration. We note that this requires the policy value to always be positive. We therefore have

$$8.007432P = 3281.247 + 23.25621 - 0.04087781P$$
$$7.966554P = 3304.50321$$
$$P = \$414.80$$

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We have

$${}_tp_x^{(00)} = e^{-\int_0^t 0.0023+0.00001(x+s)\, ds} = e^{-((0.0023+0.00001x)t+0.000005t^2)}$$

In particular,

$${}_tp_{29}^{(00)} = e^{-(0.00259t+0.000005t^2)}$$

We therefore calculate

$$\bar{a}_{29:\overline{10}|}^{(00)} = \int_0^{10} e^{-0.05t} e^{-(0.00259t+0.000005t^2)}\, dt$$

$$= \int_0^{10} e^{-(0.05259t+0.000005t^2)}\, dt$$

$$= e^{\frac{5259^2}{200000}}\int_0^{10} e^{-\frac{(t+5259)^2}{200000}}\, dt$$

$$= e^{\frac{5259^2}{200000}}100\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right) - \Phi\left(\frac{52.59}{\sqrt{10}}\right)\right)$$

$$= 7.775573$$

On the other hand we have

$$\overline{A}^{(01)}_{29:\overline{10}|} = \int_0^{10} 0.0003e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\, dt$$

$$= 0.0003\int_0^{10} e^{-(0.05259t+0.000005t^2)}\, dt$$

$$= 0.0003e^{\frac{5259^2}{200000}}\int_0^{10} e^{-\frac{(t+5259)^2}{200000}}\, dt$$

$$= 0.0003e^{\frac{5259^2}{200000}}100\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right)$$

$$= 0.002332672$$

and

$$\overline{A}^{(02)}_{29:\overline{10}|} = \int_0^{10} 0.00002(29+t)e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\, dt$$

$$= \int_0^{10} 0.00002(29+t)e^{-(0.05259t+0.000005t^2)}\, dt$$

$$= 0.00002e^{\frac{5259^2}{200000}}\int_0^{10} (t+5259-5230)e^{-\frac{(t+5259)^2}{200000}}\, dt$$

$$= 0.00002e^{\frac{5259^2}{200000}}\left(100000\left(e^{-\frac{(5259)^2}{200000}}-e^{-\frac{(5269)^2}{200000}}\right)-523000\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right)\right)$$

$$= 0.005219487$$

The EPV of benefits is therefore

$$100000 \times 0.005219487 + 200000 \times 0.002332672 = 988.48$$

and the premium is

$$\frac{988.48}{7.775573} = \$127.13$$

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$$\frac{56.94+56.61}{9878.44} = 0.01149473$$

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We calculate

$$A^{(02)\ 1}_{40:\overline{5}|} = \frac{0.30(1.03)^{-1} + 0.29(1.03)^{-2} + 0.28(1.03)^{-3} + 0.27(1.03)^{-4} + 0.27(1.03)^{-5}}{10000} = 0.000129365$$

$$A^{(03)\ 1}_{40:\overline{5}|} = \frac{1.62(1.03)^{-1} + 1.70(1.03)^{-2} + 1.78(1.03)^{-3} + 1.89(1.03)^{-4} + 1.98(1.03)^{-5}}{10000} = 0.0008191387$$

$$\ddot{a}_{40:\overline{5}|} = 1 + 0.993908(1.03)^{-1} + 0.987844(1.03)^{-2} + 0.981806(1.03)^{-3} + 0.975795(1.03)^{-4} = 4.66157$$

So the net premium is $\frac{200000 \times 0.000129365 + 100000 \times 0.0008191387}{4.66157} = \$23.12$

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(a) Using UDD in the multiple decrement table, the probability that the policy is still in force at age 46 years 3 months is $\frac{\frac{1}{4}9579.02 + \frac{3}{4}0.963844}{\frac{1}{3}10000 + \frac{2}{3}9939.08} = 0.9662829$.

The probability that the life then dies in an accident before age 47 is $\frac{3}{4} \times \frac{0.25}{\frac{1}{4} \times 9579.02 + \frac{3}{4} \times 9638.44} = 0.00001948338$.

The probability that the life dies in an accident between ages 46 and 3 months and 47 is therefore $0.9662829 \times 0.00001948338 = 0.00001882646$.

The probability that the life survives to age 47 is $\frac{9579.02}{\frac{1}{3}10000 + \frac{2}{3}9939.08} = 0.9618082$.

The probability that a life aged 47 dies in an accident before age 47 and 5 months is $\frac{5}{12} \times \frac{0.24}{9579.02}$. The probability that the life dies in an accident between ages 47 and 47 and 5 months is therefore $\frac{5}{12} \times \frac{0.24}{9579.02} \times \frac{9579.02}{\frac{1}{3}10000 + \frac{2}{3}9939.08} = 0.00001004078$.

The total probability is therefore

$0.00001882646 + 0.00001004078 = 0.00002886724$.

(b)

Using constant transition intensities, the rate of decrement for a life aged 42–43 is $-\log\left(\frac{9939.08}{10000}\right) = 0.006110632$. The probability that a policy starting at age 42 is still in force at age 42 and 4 months is $e^{-\frac{0.006110632}{3}} = 0.9979652$. The intensity of decrement aged 46–47 is $-\log\left(\frac{9579.02}{9638.44}\right) = 0.00618398$. This gives that $l^{(0)}_{46.25} = 9638.44 e^{-0.25 \times 0.00618398} = 9623.55$. The probability that the policy is still in force at age 46 and 3 months is therefore $\frac{9623.55}{9979.65} = 0.9643174$. The intensity of accidental deaths between ages 46–47 is

$$\frac{0.25}{9638.44 \int_0^1 e^{-0.00618398x}\,dx}$$

and the probability of accidental death between ages 46 and 3 months and 47 is therefore

$$\frac{0.25}{9638.44 \int_0^1 e^{-0.00618398x}\,dx} \int_0^{0.75} e^{-0.00618398x}\,dx = \frac{0.25}{9638.44} \frac{1 - e^{-0.00618398 \times 0.75}}{1 - e^{-0.00618398}} = 0.00001946839$$

The probability of surviving to age 47 is $\frac{9579.02}{9979.65} = 0.9598551$.

The intensity of decrements between ages 47–48 is
$-\log\left(\frac{9519.81}{9579.02}\right) = 0.0062004$.

The intensity of accidental deaths between ages 47–48 is

$$\frac{0.24}{9579.02 \int_0^1 e^{-0.0062004x}\,dx}$$

and the probability of accidental death between ages 47 and 47 and 5 months is therefore

$$\frac{0.24}{9579.02\int_0^1 e^{-0.0062004 0.00618398x}\,dx}\int_0^{\frac{5}{12}} e^{-0.0062004x}\,dx = \frac{0.24}{9579.02}\frac{1-e^{-0.0062004\times\frac{5}{12}}}{1-e^{-0.0062004}} = 0.00001045836$$

The total probability of dying in an accident between ages 46 years 3 months and 47 years 5 months is therefore

$$0.00001946839 \times 0.9643174 + 0.00001045836 \times 0.9598551 = 0.00002881222$$

8.10 Constructing a Multiple Decrement Table

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(a) Under UDD, suppose there are x policies at the start of the year, and during the year, y surrender and z die. Assuming UDD, the number of policies still in force at time t is $x-(y+z)t$, so the rate of dying is $\frac{z}{x-(y+z)t}$. Assume that the deaths in the new table are uniformly distributed over the year. (This is inconsistent with our uniform distribution assumption for policies with surrender, but what the hell.) If the updated mortality table has w deaths from v lives, then the rate of death at time t is $\frac{v}{w-vt}$. With this update, let x_t be the number of policies still in force at time t. We have

$$\frac{dx_t}{dt} = -\left(\frac{y}{x-(y+z)t} + \frac{v}{w-vt}\right)x_t$$

This gives

$$x_1 = xe^{-\int_0^1 \frac{y}{x-(y+z)t}+\frac{v}{w-vt}\,dt} = e^{[y\log(x-(y+z)t)+v\log(w-vt)]_0^1} = e^{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)+\log\left(\frac{w-v}{w}\right)}$$

$$= \left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)$$

In the new table, we get

$$\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}} = \left(\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)\right)^{\frac{y'}{y'+z'}}$$

$$\frac{y'}{y'+z'} = \frac{y}{y+z}\left(\frac{\log\left(\frac{x-(y+z)}{x}\right)}{\log\left(\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)\right)}\right)$$

$$= \frac{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)}{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right) + \log\left(\frac{w-v}{w}\right)}$$

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Using this, we can calculate the probabilities on the following slide.

(b) Suppose there are x policies at the start of the year, and during the year, y surrender and z die. Under constant transition probabilities, we have the number of policies still in force at time t is $xe^{\log\left(\frac{x-y-z}{x}\right)t} = x\left(\frac{x-y-z}{x}\right)^t$. If the constant rate of surrender is $\mu^{(01)}$, then

$$y = \int_0^1 \mu^{(01)} xe^{\log\left(\frac{x-y-z}{x}\right)t}\, dt = \mu^{(01)} x \frac{1 - \frac{x-y-z}{x}}{\log(x) - \log(x-y-z)} = \mu^{(01)} \frac{y+z}{\log(x) - \log(x-y-z)}$$

This gives $\mu^{(01)} = \frac{y(\log(x) - \log(x-y-z))}{(y+z)}$, so without deaths, the probability of surrender is $e^{-(\log(x) - \log(x-y-z))\frac{y}{(y+z)}} = \left(\frac{x-y-z}{x}\right)^{\frac{y}{y+z}}$. The table is therefore the same as in part (a).

(c) If each independent decrement satisfies UDD, then suppose the probabilities for surrender and death as only decrements are p and q respectively, then the rate of surrender is $\frac{p}{1-pt}$ and the rate of death is $\frac{q}{1-qt}$.

Now in a model with two decrements, the total rate of decrement is $\frac{p}{1-pt} + \frac{q}{1-qt}$, so the total probability of no decrement in the year is

$$e^{-\int_0^1 \frac{p}{1-pt} + \frac{q}{1-qt}\, dt} = e^{[\log(1-pt) + \log(1-qt)]_0^1} = (1-p)(1-q)$$

Similarly, the probability of no decrement before time t is $(1 - pt)(1 - qt)$. The probability of surrender is

$$\int_0^1 (1-pt)(1-qt)\frac{p}{1-pt}\, dt = \int_0^1 p(1-qt)\, dt = p\left(1 - \frac{q}{2}\right)$$

Similarly, the probability of death is $q\left(1 - \frac{p}{2}\right)$.

If we are given the probabilities of surrender and death in the multiple decrement model are a and b respectively, then we have to solve

$$p\left(1 - \frac{q}{2}\right) = a$$

$$q\left(1 - \frac{p}{2}\right) = b$$

$$p - q = a - b$$

$$p - \frac{p^2}{2} = a - \frac{p}{2}(a - b)$$

$$p^2 + (b - a - 2)p + 2a = 0$$

$$p = \frac{a + 2 - b \pm \sqrt{(a + 2 - b)^2 - 8a}}{2}$$

$$= \frac{a + 2 - b \pm \sqrt{a^2 + b^2 + 4 - 2ab - 4a - 4b}}{2}$$

$$q = \frac{b + 2 - a \pm \sqrt{a^2 + b^2 + 4 - 2ab - 4a - 4b}}{2}$$

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x	l_x	d_x
40	10000.00	59.01
41	9940.99	59.02
42	9881.98	59.03
43	9822.95	59.04
44	9763.91	59.06
45	9704.85	59.07
46	9645.78	59.08
47	9586.69	59.11

9.2 Joint Life and Last Survivor Benefits

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Advantages	Disadvantages
Annuity value does not depend on time of death	Value of benefit varies with time of death

9.4 Independent Future Lifetimes

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$$\ddot{a}_{63,62:\overline{10}|} = 1 + 0.999830\frac{9996.47}{9998.30}(1.07)^{-1} + 0.999647\frac{999449}{9998.30}(1.07)^{-2} + 0.999449\frac{9992.35}{9998.30}(1.07)^{-3} + 0.999235\frac{9990.03}{9998.30}(1.07)^{-4}$$
$$+ 0.999003\frac{9987.53}{9998.30}(1.07)^{-5} + 0.998753\frac{9984.83}{9998.30}(1.07)^{-6} + 0.998483\frac{9981.91}{9998.30}(1.07)^{-7} + 0.998191\frac{9978.76}{9998.30}(1.07)^{-8}$$
$$+ 0.997876\frac{9975.34}{9998.30}(1.07)^{-9}$$
$$= 7.502352$$

$$\ddot{a}_{\overline{63,62:10|}} = 0.0000170 \times \frac{1.83}{9998.30}(1.07)^{-1} + \left(0.0000353 \times \frac{1.83 + 1.98}{9998.30} - 0.0000170 \times \frac{1.83}{9998.30}\right)(1.07)^{-2}$$

$$+ \left(0.0000551 \times \frac{1.83 + 1.98 + 2.14}{9998.30} - 0.0000353 \times \frac{1.83 + 1.98}{9998.30}\right)(1.07)^{-3}$$

$$+ \left(0.0000765 \times \frac{1.83 + 1.98 + 2.14 + 2.31}{9998.30} - 0.0000551 \times \frac{1.83 + 1.98 + 2.14}{9998.30}\right)(1.07)^{-4}$$

$$+ \left(0.0000996 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50}{9998.30} - 0.0000765 \times \frac{1.83 + 1.98 + 2.14 + 2.31}{9998.30}\right)(1.07)^{-5}$$

$$+ \left(0.0001246 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70}{9998.30} - 0.0000996 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50}{9998.30}\right)(1.07)^{-6}$$

$$+ \left(0.0001516 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92}{9998.30}\right.$$
$$\left. -0.0001246 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70}{9998.30}\right)(1.07)^{-7}$$

$$+ \left(0.0001808 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92 + 3.16}{9998.30}\right.$$
$$\left. -0.0001516 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92}{9998.30}\right)(1.07)^{-8}$$

$$+ \left(0.0002124 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92 + 3.16 + 3.41}{9998.30}\right.$$
$$\left. -0.0001808 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92 + 3.16}{9998.30}\right)(1.07)^{-9}$$

$$+ \left(0.0002465 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92 + 3.16 + 3.41 + 3.69}{9998.30}\right.$$
$$\left. -0.0002124 \times \frac{1.83 + 1.98 + 2.14 + 2.31 + 2.50 + 2.70 + 2.92 + 3.16 + 3.41}{9998.30}\right)(1.07)^{-10}$$

$$= 0.0000003531257$$

So the premium is $\frac{2000000 \times 0.0000003531257}{7.502352} = \$0.09$.

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If the Husband is dead and the wife is alive at the end of the policy with probability 0.997567×0.001251, then the wife receives a reversionary annuity with value $30000 \times 13.89755 = 416926.5$. The expected present value of payments after the end of the policy is therefore $30000 \times 13.89755 \times 0.997567 \times 0.001251(1.05)^{-7} = 369.7718$. Payments during the term of the policy are received if and only if the husband is dead and the wife is alive. The expected present value of payments received during the term of the policy is therefore

$$30000 \left(0.000135 \times 0.999736(1.05)^{-1} + 0.000283 \times 0.999448(1.05)^{-2} + 0.000444 \times 0.999134(1.05)^{-3}\right.$$
$$\left. +0.000620 \times 0.998791(1.05)^{-4} + 0.000812 \times 0.998419(1.05)^{-5} + 0.001022 \times 0.998012(1.05)^{-6}\right) = 80.22267$$

So the total expected benefit is $369.7718 + 80.22267 = 449.9945$.

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For the premiums, we have

$$\ddot{a}_{53,64:\overline{7}|} = 1+0.999865\times0.999736(1.05)^{-1}+0.999717\times0.999448(1.05)^{-2}+0.999555\times0.999134(1.05)^{-3}+0.999380\times0.998791(1.05)^{-4}+0$$

So the net annual premium is $\frac{80.22267}{6.067799} = \$13.22$.

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The probabilities of each being dead after n years are:

Years	P(Husband Dead)	P(Wife Dead)	P(Survivor)	EPV benefit
0	0.000000	0.000000	1.0000000	1.0000000
1	0.057684	0.000103	0.9999941	0.9433906
2	0.120792	0.000213	0.9999743	0.8899735
3	0.189153	0.000331	0.9999374	0.8395667
4	0.262348	0.000457	0.9998801	0.7919987
5	0.339656	0.000592	0.9997989	0.7471079
6	0.420004	0.000736	0.9996909	0.7047426
7	0.501937	0.000891	0.9995528	0.6647597
8	0.583624	0.001057	0.9993831	0.6270253
9	0.662908	0.001236	0.9991806	0.5914135
10	0.737429	0.001428	0.9989470	0.5578068
11	0.804821	0.001635	0.9986841	0.5260943
12	0.862981	0.001858	0.9983966	0.4961725
13	0.910384	0.002098	0.9980900	0.4679435
14	0.946379	0.002357	0.9977694	0.4413144
15	0.971384	0.002637	0.9974385	0.4161962
16	0.986877	0.002939	0.9970996	0.3925045
17	0.995126	0.003265	0.9967509	0.3701578
18	0.998683	0.003617	0.9963878	0.3490783
19	0.999799	0.003998	0.9960028	0.3291919
total				12.14644

The EPV of benefits after 20 years is $0.9955900(1.06)^{-20} \times 16.1807 = 5.022969$, so total EPV is $12.14644 + 5.022969 = 17.16941$.

Net premium is $45000 \times 17.16941 = \$772,623.45$.

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Year	P(Husband Dies)	P(Wife Dies)	P(Both die)	P(One dies)
1	0.000180	0.001651	0.00000029718	0.001830994
2	0.000193	0.001785	0.00000035343	0.001977993
3	0.000208	0.001929	0.000000401232	0.002136992
4	0.000223	0.002085	0.000000464955	0.002307991
5	0.000240	0.002254	0.00000054096	0.002493989
6	0.000258	0.002435	0.00000062823	0.002692987
7	0.000278	0.002631	0.000000731418	0.002908985

Annual EPV from one dying: 0.0138451

Annual EPV from both dying 0.000002863602

$x = (1+i)^{\frac{1}{12}}$

$$y = \frac{23}{144}x^{11} + \frac{21}{144}x^{10} + \frac{19}{144}x^9 + \frac{17}{144}x^8 + \frac{15}{144}x^7 + \frac{13}{144}x^6 + \frac{11}{144}x^5 + \frac{9}{144}x^4 + \frac{7}{144}x^3 + \frac{5}{144}x^2 + \frac{3}{144}x + \frac{1}{144}$$

$$y - xy = \frac{1}{144} + \frac{2}{144}(x + x^2 + \cdots + x^{11}) - \frac{23}{144}x^{12}$$

$$= \frac{1}{144} + \frac{2x}{144}\left(\frac{x^{11}-1}{x-1}\right) - \frac{23}{144}x^{12}$$

$$y = \frac{1}{144(x-1)}\left(23x^{12} - 1 - 2\frac{x^{11}-1}{x-1}\right)$$

$$= \frac{1}{12i^{(12)}}\left(23i + 22 - 2\frac{x^{11}-1}{x-1}\right)$$

$i^{(12)} = 0.039288$
$y = 1.024815$

$$A^{(12\ 1)}_{45,76:\overline{7}|} = 0.0138451\frac{i}{i^{(12)}} + 0.000002863602 \times 1.024815 = 0.01410006$$

$$A^{(12)}_{45,76:\overline{7}|} = 0.01410006 + 0.997090283582(1.04)^{-7} = 0.7718067$$

$$d^{(12)} = 12\left(1 - (1.04)^{-\frac{1}{12}}\right) = 0.03915669$$

This gives $\ddot{a}^{(12)}_{45,76:\overline{7}|} = \frac{1-0.7718067}{0.03915669} = 5.827696$

So the monthly premiums are $\frac{850000 \times 0.01410006}{12 \times 5.827696} = \$171.38$

9.6 A Model with Dependent Future Lifetimes

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(a)

Year	P(Husband Dies)	P(Wife Survives)	P(Wife Survives to 49)	P(Husband Dies and Wife Survives to 49)
1	0.004599	0.999900	$\frac{9961.92}{9997.47}$	0.004582188
2	0.004984	0.999794	$\frac{9961.92}{9994.72}$	0.004966621
3	0.005398	0.999681	$\frac{9961.92}{9991.74}$	0.005380173
4	0.005844	0.999560	$\frac{9961.92}{9988.49}$	0.00582589
5	0.006324	0.999432	$\frac{9961.92}{9984.95}$	0.00630583
6	0.006840	0.999294	$\frac{9961.92}{9981.10}$	0.006822036
7	0.007394	0.999148	$\frac{9961.92}{9976.91}$	0.007376601
8	0.007989	0.998991	$\frac{9961.92}{9972.34}$	0.0079726
9	0.008625	0.998823	$\frac{9961.92}{9967.36}$	0.008610147
$\geqslant 10$	0.942003	0.998644	1	0.9407256
				0.9985677

(b)

Suppose the probability of the wife surviving while the husband is alive is p_1 and while the husband is dead is p_2. If the husband dies at time t during the year, the wife's probability of surviving the year is $(1 - tq_1)\left(\frac{p_2}{1-tq_2}\right)$

The probability of the wife's surviving the year if the husband dies at a uniformly distributed time during the year is therefore

$$\int_0^1 (1 - tq_1)\left(\frac{p_2}{1 - tq_2}\right)dt = p_2\left(\frac{q_1}{q_2} + (q_2 - q_1)\log(1 - q_2)\right)$$

Year	P(Husband Dies)	P(Wife Survives to start of year)	P(Wife survives year given Husband dies)	P(Wife Survives to 49)	P(Husband Dies and Wife Survives to 49)
1	0.004599	1	0.9998235	$\frac{9961.92}{9997.47}$	0.004581838
2	0.004984	0.999900	0.9995565	$\frac{9961.92}{9994.72}$	0.004964944
3	0.005398	0.999794	0.9992660	$\frac{9961.92}{9991.74}$	0.005376837
4	0.005844	0.999681	0.9989505	$\frac{9961.92}{9988.49}$	0.005820486
5	0.006324	0.999560	0.9986070	$\frac{9961.92}{9984.95}$	0.006297865
6	0.006840	0.999432	0.9982330	$\frac{9961.92}{9981.10}$	0.006810936
7	0.007394	0.999294	0.9978261	$\frac{9961.92}{9976.91}$	0.007361662
8	0.007989	0.999148	0.9973831	$\frac{9961.92}{9972.34}$	0.007953002
9	0.008625	0.998991	0.9969001	$\frac{9961.92}{9967.36}$	0.008584917
10	0.009305	0.998823	0.9963737	1	0.009260354
$\geqslant 10$	0.932698	0.998644	1	1	0.9407256
					0.9984461

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By summing over times the husband dies, we calculate the following lifetable for the wife.

Suppose the mortality for the wife while the husband is alive is q_1, the mortality while the husband is dead is q_2, and the mortality of the husband is q_3. Conditional on the husband dying at time t in the year, the probability that the wife dies during the year is

$$tq_1 + (1 - tq_1)\left(1 - \frac{1 - q_2}{1 - tq_2}\right) = tq_1 + (1 - tq_1)\frac{(1 - t)q_2}{1 - tq_2}$$

The total probability that the wife dies during the year if the husband is alive at the start of the year is therefore

$$(1 - q_3)q_1 + \int_0^1 q_3 \left(tq_1 + (1 - tq_1)\frac{(1-t)q_2}{1 - tq_2} \right) dt$$

$$= (1 - q_3)q_1 + \frac{q_1 q_3}{2} + q_2 q_3 \int_0^1 (1-t)\frac{1 - tq_1}{1 - tq_2} dt$$

$$= (1 - q_3)q_1 + \frac{q_1 q_3}{2} + q_2 q_3 \int_0^1 \frac{1 - (1+q_1)t + q_1 t^2}{1 - tq_2} dt$$

$$= (1 - q_3)q_1 + \frac{q_1 q_3}{2} + q_2 q_3 \int_0^1 \frac{1 + q_1 + \frac{q_1}{q_2}}{q_2} - \frac{q_1}{q_2}t + \frac{1 - \frac{1+q_1+\frac{q_1}{q_2}}{q_2}}{1 - tq_2} dt$$

$$= (1 - q_3)q_1 + \frac{q_1 q_3}{2} + q_2 q_3 \left(\frac{1 + q_1 + \frac{q_1}{q_2}}{q_2} - \frac{q_1}{2q_2} + \left(1 - \frac{1 + q_1 + \frac{q_1}{q_2}}{q_2} \right) \left[-\frac{\log(1 - tq_2)}{q_2} \right]_0^1 \right)$$

$$= (1 - q_3)q_1 + q_3 \left(1 + q_1 + \frac{q_1}{q_2} \right) + q_3 \left(1 - \frac{1 + q_1 + \frac{q_1}{q_2}}{q_2} \right) (\log(1) - \log(1 - q_2))$$

$$= (1 - q_3)q_1 + q_3 \left(1 + q_1 + \frac{q_1}{q_2} \right) - q_3 \left(\frac{q_2^2 - q_2 - q_1 q_2 - q_1}{q_2^2} \right) \log(1 - q_2)$$

$$= q_1 + q_3 \left(\frac{q_1 + q_2}{q_2} \right) + q_3 \left(\frac{q_2 + q_1 q_2 + q_1 - q_2^2}{q_2^2} \right) \log(1 - q_2)$$

If the probability that the husband and wife are both alive at the start of the year is p_1 and the probability that the wife is alive, but the husband is dead is p_2, then the probability that both are alive at the end of the year is $p_1(1 - q_1)(1 - q_3)$. The probability that the wife is alive and the husband is dead at the end of the year is therefore:

$$p_2(1 - q_2) + p_1 \left(1 - \left(q_1 + q_3 \left(\frac{q_1 + q_2}{q_2} \right) + q_3 \left(\frac{q_2 + q_1 q_2 + q_1 - q_2^2}{q_2^2} \right) \log(1 - q_2) \right) - (1 - q_1)(1 - q_3) \right)$$

$$= p_2(1 - q_2) + p_1 \left(q_3(1 - q_1) - \left(q_3 \left(\frac{q_1 + q_2}{q_2} \right) + q_3 \left(\frac{q_2 + q_1 q_2 + q_1 - q_2^2}{q_2^2} \right) \log(1 - q_2) \right) \right)$$

$$= p_2(1 - q_2) - p_1 q_3 \left(\left(\left(q_1 + \frac{q_1}{q_2} \right) + \left(\frac{q_2 + q_1 q_2 + q_1 - q_2^2}{q_2^2} \right) \log(1 - q_2) \right) \right)$$

Year	Both Alive	Husband Dead, Wife Alive	Wife Alive
0	10000.00	0.00	10000.00
1	9953.01	45.99	9999.00
2	9902.13	95.80	9997.93
3	9847.05	149.74	9996.79
4	9787.45	208.11	9995.56
5	9722.99	271.24	9994.23
6	9653.31	339.49	9992.80
7	9578.01	413.22	9991.23
8	9496.70	492.84	9989.54
9	9408.95	578.74	9987.70
10	9314.33	671.35	9985.69

This gives

$$a_{39:\overline{10}|} = 8.431041$$

$$A_{39:\overline{10}|} = 0.001132052$$

So the annual premium is

$$\frac{200000 \times 0.001132052}{8.431041} = \$26.85$$

9.7 The Common Shock Model

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We have

$$_tp_{25:56}^{(00)} = e^{-\int_0^t 0.000042+0.000000001(56+s)+0.000000002(25+s)+0.000001(25+s)^2+0.000002(56+s)^2 ds}$$

$$= e^{-\int_0^t 0.003403106+0.000174003s+0.000003s^2 ds}$$

$$= e^{-(0.003403106t+0.0000870015t^2+0.000001t^3)}$$

So the probability is given by

$$\int_0^{10} e^{-(0.003403106t+0.0000870015t^2+0.000001t^3)}(0.000002(25+t)^2 + 0.000000002(56+t))e^{-0.000002\frac{(66^3-(56+t)^3)}{3}} dt$$

$$= \int_0^{10} e^{-(0.07458667-0.002868894t-0.0000249985t^2-0.000001t^3)}(0.001250112 + 0.000100002t + 0.000002t^2)dt$$

$$= 0.01715084$$

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If the husband dies first after time t, then the present value (at time of husband's death) of the life annuity is

$$25000\int_0^{\infty} e^{-0.005s}e^{-0.04s}\,ds = \frac{25000}{0.045} = 555555.56$$

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The total transition intensity out of state 0 is

$$(0.000001(56+t)^2 + 0.000000001(25+t)) + (0.000002(25+t)^2 + 0.000000002(56+t)) + 0.000042$$
$$= 0.000003t^2 + 0.000212003t + 0.004428137$$

The expected value of the life annuity is therefore

$$555555.56 \int_0^\infty (0.000002(25+t)^2 + 0.000000002(56+t))e^{-(0.000003t^2+0.000212003t+0.004428137)}e^{-0.04t}\, dt$$
$$= 555555.56 \int_0^\infty (0.000002(25+t)^2 + 0.000000002(56+t))e^{-(0.000003t^2+0.040212003t+0.004428137)}\, dt$$
$$= \$84,251.58$$

(Numerically integrated)
The expected present value of the premiums is

$$P \int_0^\infty e^{-(0.000003t^2+0.000212003t+0.004428137)}e^{-0.04t}\, dt$$
$$= P \int_0^\infty e^{-(0.000003t^2+0.040212003t+0.004428137)}\, dt$$
$$= P \int_0^\infty e^{-(0.000003(t+6702.0005)^2-134.746)}\, dt$$
$$= Pe^{134.746} \int_0^\infty e^{-0.000003(t+6702.0005)^2}\, dt$$
$$= Pe^{134.746} \sqrt{\frac{\pi}{0.000003}} \left(1 - \Phi(6702.0005\sqrt{0.000006})\right)$$
$$= 24.66746P$$

We therefore get that

$$24.66746P = 84251.58$$
$$P = \$3,415.49$$

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We have

$${}_tp_{xy}^{(00)} = e^{-\int_0^t 0.001002(y+s)+0.002001(x+s)+0.012\, ds} = e^{-((0.001002y+0.002001x+0.012)s+0.0015015s^2)}$$

Numerically integrating gives the following lifetable

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Year	Both alive	H alive W dead	H dead W alive	Both dead	Year	Both alive	H alive W dead	H dead W alive	Both dead
0	10000.00	0.00	0.00	0.00	45	0.09	0.12	113.41	9886.39
1	8247.83	239.88	1332.93	179.36	46	0.06	0.09	97.73	9902.13
2	6782.28	394.62	2365.22	457.88	47	0.05	0.06	84.04	9915.85
3	5560.41	485.42	3144.19	809.98	48	0.03	0.05	72.13	9927.79
4	4545.01	529.17	3711.21	1214.61	49	0.02	0.03	61.78	9938.17
5	3703.89	539.19	4102.32	1654.61	50	0.02	0.02	52.81	9947.15
6	3009.38	525.84	4348.71	2116.08	51	0.01	0.02	45.05	9954.92
7	2437.76	497.08	4477.26	2587.90	52	0.01	0.01	38.35	9961.63
8	1968.80	458.92	4511.01	3061.27	53	0.01	0.01	32.58	9967.40
9	1585.29	415.81	4469.58	3529.31	54	0.00	0.01	27.63	9972.36
10	1272.65	370.99	4369.61	3986.74	55	0.00	0.00	23.38	9976.61
11	1018.61	326.71	4225.09	4429.59	56	0.00	0.00	19.74	9980.25
12	812.84	284.47	4047.71	4854.98	57	0.00	0.00	16.64	9983.35
13	646.68	245.24	3847.19	5260.88	58	0.00	0.00	14.00	9986.00
14	512.95	209.54	3631.53	5645.98	59	0.00	0.00	11.75	9988.25
15	405.66	177.59	3407.22	6009.54	60	0.00	0.00	9.84	9990.15
16	319.84	149.39	3179.52	6351.25	61	0.00	0.00	8.23	9991.77
17	251.43	124.80	2952.61	6671.16	62	0.00	0.00	6.87	9993.13
18	197.05	103.58	2729.76	6969.60	63	0.00	0.00	5.72	9994.28
19	153.97	85.45	2513.48	7247.10	64	0.00	0.00	4.75	9995.25
20	119.95	70.09	2305.62	7504.34	65	0.00	0.00	3.94	9996.06
21	93.17	57.17	2107.54	7742.12	66	0.00	0.00	3.26	9996.74
22	72.15	46.39	1920.14	7961.32	67	0.00	0.00	2.70	9997.30
23	55.70	37.45	1743.97	8162.88	68	0.00	0.00	2.22	9997.78
24	42.87	30.09	1579.29	8347.75	69	0.00	0.00	1.83	9998.17
25	32.90	24.06	1426.12	8516.91	70	0.00	0.00	1.50	9998.50
26	25.18	19.15	1284.33	8671.34	71	0.00	0.00	1.23	9998.77
27	19.20	15.17	1153.63	8812.00	72	0.00	0.00	1.01	9998.99
28	14.61	11.97	1033.61	8939.81	73	0.00	0.00	0.82	9999.18
29	11.08	9.40	923.82	9055.71	74	0.00	0.00	0.67	9999.33
30	8.37	7.35	823.71	9160.56	75	0.00	0.00	0.54	9999.46
31	6.31	5.73	732.75	9255.21	76	0.00	0.00	0.44	9999.56
32	4.74	4.44	650.34	9340.48	77	0.00	0.00	0.36	9999.64
33	3.55	3.43	575.90	9417.11	78	0.00	0.00	0.29	9999.71
34	2.65	2.64	508.86	9485.85	79	0.00	0.00	0.23	9999.77
35	1.98	2.03	448.64	9547.36	80	0.00	0.00	0.19	9999.81
36	1.47	1.55	394.69	9602.29	81	0.00	0.00	0.15	9999.85
37	1.09	1.18	346.50	9651.24	82	0.00	0.00	0.12	9999.88
38	0.80	0.89	303.54	9694.76	83	0.00	0.00	0.10	9999.90
39	0.59	0.67	265.36	9733.38	84	0.00	0.00	0.08	9999.92
40	0.43	0.51	231.49	9767.57	85	0.00	0.00	0.06	9999.94
41	0.32	0.38	201.53	9797.77	86	0.00	0.00	0.05	9999.95
42	0.23	0.28	175.08	9824.40	87	0.00	0.00	0.04	9999.96
43	0.17	0.21	151.80	9847.82	88	0.00	0.00	0.03	9999.97
44	0.12	0.16	131.34	9868.38	89	0.00	0.00	0.02	9999.98

(a) Summing up we get that $A_{\overline{75,29}} = 0.4811416$. This gives $a_{\overline{75,29}} = \frac{1-0.4811416}{0.06} = 8.64764$. so the premium is $300000 \times 0.48114168.64764 = \$16,691.55$.

(b) After 10 years, if the husband is dead, but the wife is alive, then the mortality is $\mu_y^{23} = 0.002y$. The probability that she survives for s years is therefore

$$e^{-\int_0^s 0.002(39+s)\,ds} = e^{-0.078s-0.001s^2}$$

This gives $A_{39} = \sum_{i=1}^{\infty}(1.06)^{-i}(e^{-0.078(i-1)-0.001(i-1)^2} - e^{-0.078i-0.001i^2}) = 0.590609$, so $\ddot{a}_{39} = \frac{1-0.590609}{0.06} = 6.823183$. This means that the policy value is

$$300000 \times 0.590609 - 16691.55 \times 6.823183 = \$63,293.20$$

The Salary Scale Function

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(a) Average Salary from age 62–65 is given by $\frac{60000(1.03^{20}+1.03^{21}+1.03^{22})}{3} = \$111,650.18$

(b) Average Salary from age 62–65 is given by $\frac{60000(2.261+2.335+2.400)}{3} = \$139,920$

(c) for scale in (a) we have $\frac{60000(1.03^{19.6666667}+1.03^{20.6666667}+1.03^{21.6666667})}{3} = \$110,445.50$

for scale in (b), we can use linear interpolation to estimate $s_{42.33333333} = \frac{2}{3} \times 1 + \frac{1}{3} \times 1.036 = 1.012$. This gives a final average salary of $\frac{60000(2.261+2.335+2.400)}{3\times 1.012} = \$138,260.87$

10.4 Setting the DC Contribution

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If current salary is 1, final average salary is $\frac{1.03^{32}+1.03^{33}+1.03^{34}}{3} = 2.653108$. The replacement ratio means the original annuity is worth 1.591865, and the reversionary annuity is worth 0.7959323.

$$_tp_{65} = e^{-\int_0^t 0.000002e^{\log(1.093)(65+t)}dt} = e^{-0.0006476502\left[\frac{e^{\log(1.093)s}}{\log(1.093)}\right]_0^t} = e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}$$

The value of the life annuity is given by

$$\overline{a}_{65} = \int_0^{\infty} e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 21.07607$$

The value of the reversionary annuity is given by

$$\overline{a}_{65|62} = \int_0^{\infty}\left(1 - e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 1.416998$$

So the EPV of the benefits at the time of retirement is

$$1.591865 \times 21.07607 + 0.7959323 \times 1.416998 = 34.67809$$

If first monthly salary is x, then we have

$$x\left(1 + 1.03^{\frac{1}{12}} + \ldots + 1.03^{\frac{11}{12}}\right) = 1$$

$$x\frac{1.03 - 1}{1.03^{\frac{1}{12}} - 1} = 1$$

$$x = \frac{1.03^{\frac{1}{12}} - 1}{0.03} = 0.08220899$$

The accumulated value of all salary paid monthly in arrear at the end of 35 years is

$$0.08220899\sum_{i=1}^{420}(1.03)^{\frac{i}{12}}(1.06)^{35-\frac{i}{12}} = 0.08220899(1.03)^{35}\sum_{i=1}^{420}\left(\frac{1.06}{1.03}\right)^{\frac{i}{12}} = 0.08220899(1.03)^{35}\frac{\left(\frac{1.06}{1.03}\right)^{35} - 1}{\left(\frac{1.06}{1.03}\right)^{\frac{1}{12}} - 1} = 167.2143$$

So the percentage of salary needed each month is

$$\frac{34.67809}{167.2143} = 20.74\%$$

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For questions (a), (b), (e), (f) and (g), the employee's accumulated total is still 34.67809, and the final average salary is 2.653108.

For (a), the replacement ratio is $\frac{34.67809}{1.591865 \times 21.07607} \times 60\% = 62.02\%$

For (b), the value of the reversionary annuity is

$$\bar{a}_{65|62} = \int_0^\infty \left(1 - e^{-0.007283007\left(e^{\log(1.093)t} - 1\right)}\right)e^{-0.01483447\left(e^{\log(1.093)t} - 1\right)}e^{-log(1.04)t}dt = 0.9445889$$

so the replacement ratio is $\frac{34.67809}{1.591865 \times 21.07607 + 0.7959323 \times 0.9445889} \times 60\% = 60.66\%$

For (e) the replacement ratio is $\frac{34.67809}{1.591865 \times 21.07607 + 0.4775595 \times 1.416998} \times 60\% = 60.79\%$

For (f) we have

$$\bar{a}_{65} = \int_0^\infty e^{-0.007283007\left(e^{\log(1.093)t} - 1\right)}e^{-log(1.03)t}dt = 25.05707$$

The value of the reversionary annuity is given by

$$\bar{a}_{65|62} = \int_0^\infty \left(1 - e^{-0.007283007\left(e^{\log(1.093)t} - 1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t} - 1\right)}e^{-log(1.04)t}dt = 2.080705$$

so the replacement ratio is $\frac{34.67809}{1.591865 \times 25.05707 + 0.7959323 \times 2.080705} \times 60\% = 50.08\%$

For (g) we have

$$\bar{a}_{65} = \int_0^\infty e^{-0.08872485\left(e^{\log(1.143)t} - 1\right)}e^{-log(1.04)t}dt = 11.24322$$

and

$$\bar{a}_{65|62} = \int_0^\infty \left(1 - e^{-0.08872485\left(e^{\log(1.143)t} - 1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t} - 1\right)}e^{-log(1.04)t}dt = 10.30211$$

so the replacement ratio is $\frac{34.67809}{1.591865 \times 11.24322 + 0.7959323 \times 10.30211} \times 60\% = 79.73\%$

For (c), the accumulated value of the investments is

$$0.08220899(1.03)^{35} \frac{\left(\frac{1.07}{1.03}\right)^{35} - 1}{\frac{1.07}{1.03} - 1} = 203.264$$

So the replacement ratio is $\frac{203.264}{167.2143} \times 60\% = 72.94\%$.

For (e), the accumulated value of the investments is

$$0.08220899(1.05)^{35} \frac{\left(\frac{1.06}{1.05}\right)^{35} - 1}{\left(\frac{1.06}{1.05}\right)^{\frac{1}{12}} - 1} = 225.7627$$

but the final average salary is $\frac{1.05^{32} + 1.05^{33} + 1.05^{34}}{3} = 5.007159$

So the new replacement ratio is $\frac{225.7627 \times 2.653108}{167.2143 \times 5.007159} \times 60\% = 42.92\%$.

10.5 The Service Table

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(a) See slide

(b) $\frac{62.28}{4882.68} = 0.01275529$

10.6 Valuation of Benefits

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(a)

$$\ddot{a}_{65}^{(12)} = \frac{1}{12} \sum_{n=0}^{\infty} (1.05)^{-\frac{n}{12}} e^{-\int_0^{\frac{n}{12}} 0.0000023(1.12)^{65+t}\, dt} = \frac{1}{12} \sum_{n=0}^{\infty} (1.05)^{-\frac{n}{12}} e^{-0.0000023(1.12)^{65} \frac{(1.12)^{\frac{n}{12}} - 1}{\log(1.12)}} = 14.18721$$

The member's final average salary is $76000 \frac{(1.04)^{16} + (1.04)^{17} + (1.04)^{18}}{3} = \$148,116.36$.

The EPV of the accrued benefit conditional on the individual retiring at age 65 is therefore:

$$148116.36 \times 13 \times 0.01 \times 14.18721(1.05)^{-18} = \$113,510.49$$

(b)

$$\ddot{a}_{60}^{(12)} = \frac{1}{12} \sum_{n=0}^{\infty} (1.05)^{-\frac{n}{12}} e^{-0.0000023(1.12)^{60} \frac{(1.12)^{\frac{n}{12}} - 1}{\log(1.12)}} = 15.38250094$$

The member's final average salary is $76000 \frac{(1.05)^{11} + (1.04)^{12} + (1.04)^{13}}{3} = \$121,740.85$.

The EPV of the accrued benefit conditional on the individual retiring at age 60 is therefore:

$$121740.85 \times 13 \times 0.01 \times 15.38250094(1.05)^{-13} = \$129,105.80$$

(c)

age	probability of retirement	EPV of benefits	Probability times EPV
60	0.3	129105.80	38731.73853
60.5	0.09750441650	127563.3999	12437.99487
61.5	0.08392282902	124466.4712	10445.57838
62.5	0.07223304834	121353.9472	8765.765534
63.5	0.06217156087	118226.6896	7350.337829
64.5	0.05351155835	115085.7472	6158.417677
65	0.3306565869	113510.49	37532.9912
total			121422.824

So the EPV of accrued benefits is \$121,422.82.

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We calculate $_tp_{65} = e^{-\int_0^t 0.000002(1.102)^{65+t}\,dt} = e^{-0.000002(1.102)^{65}\left(\frac{1.102^t-1}{\log(1.102)}\right)} = e^{-0.01136305(1.102^t-1)}$

$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty} e^{-0.01136305(1.102^{\frac{n}{12}}-1)}(1.04^{-\frac{n}{12}} = 19.69323$

(a) If he withdraws today, he receives an annual pension of $75000 \times 0.02 \times 15 \times 1.02^{22} = \$34,784.54$

The EPV of this is $34784.54 \times 19.69323 \times_{22} p_{43}(1.04)^{-22}$

We have $_{22}p_{43} = e^{-\int_0^{22} 0.000002(1.102)^{43+t}\,dt} = e^{-0.000002(1.102)^{43}\left(\frac{1.102^{22}-1}{\log(1.102)}\right)} = 0.9900282$

So the EPV is \$286,165.53

(b) If the employee withdraws after t years,then his annual salary is $75000(1.05)^t$, so his accrued withdrawl benefits have present value $75000(1.05)^t \times 0.02 \times 15(1.02)^{22-t} \times 19.69323_{22-t}p_{43+t}(1.04)^{-22}$.

Conditional on the employee withdrawing before age 60, the EPV of the accrued withdrawl benefits is

$$\int_0^{17} \frac{286165.53}{_tp_{43}}\left(\frac{1.05}{1.02}\right)^t e^{-0.07(43+t)}e^{\frac{e^{-0.07t}-e^{-0.07 times 17}}{0.07}}\,dt = \$215,505.95$$

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We calculate $_tp_{65} = e^{-\int_0^t 0.0000023(1.12)^{65+t}\,dt} = e^{-0.0000023(1.12)^{65}\left(\frac{1.12^t-1}{\log(1.12)}\right)} = e^{-0.03210402(1.12^t-1)}$

$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty} e^{-0.03210402(1.12^{\frac{n}{12}}-1)}(1.05^{-\frac{n}{12}} = 14.03364$

If the member withdraws at age x, then the salary is $45000(1.04)^{x-46}\left(\frac{1.04^{-1}+1.04^{-2}+1.04^{-3}}{3}\right)$, so with the

COLA, the accrued pension has an annual value of $41626.3655 \times 0.015 \times 13(1.04)^{x-46}(1.02)^{65-x} = 11825.14209\left(\frac{1.04}{1.02}\right)^{x-46}$, so the value at age 65 is

$$165949.787\left(\frac{1.04}{1.02}\right)^{x-46}$$

. Discounting at 5% to the age of withdrawl and at 6% to the present day, gives a conditional present value of

$$165949.787\left(\frac{1.04 \times 1.06}{1.02 \times 1.05}\right)^{x-46}(1.05)^{46-65} = 65671.96589\,(1.029318394)^{x-46}$$

The EPV is therefore $65671.96589\,(1.029318394)^{x-46}_{65-x}p_{46+x}$

We have $_{65-x}p_{46+x} = e^{-0.0000023(1.12)^{65}\left(\frac{1-1.12^{x-65}}{\log(1.12)}\right)} = e^{-0.03210402(1-1.12^{x-65})}$ so the EPV is

$$65671.96589\,(1.029318394)^{x-46}\,e^{-0.03210402(1-1.12^{x-65})}$$

The probability that the member is still enrolled in the plan at age x is $e^{-\int_{46}^x e^{-0.07y}+0.0000023(1.12)^y\,dy} = e^{-\frac{(e^{-3.22}-e^{-0.07x})}{0.07}-0.00002029494998(1.12^x-1.12^{46})}$ the EPV of accrued pension benefits paid to early withdrawls is therefore

$$\int_{46}^{60} 65671.96\,(1.02932)^{x-46}\,e^{-0.03210(1-1.12^{x-65})}e^{-0.07x}e^{-\frac{(e^{-3.22}-e^{-0.07x})}{0.07}-0.000020295(1.12^x-1.12^{46})}$$

$$= 26792.25309$$

The probability he is still employed at age 60 is $e^{-\int_{46}^{60} e^{-0.07x}+0.0000023(1.12)^x\,dx} = e^{-\frac{e^{-3.22}-e^{-4.2}}{0.07}-0.0000023\frac{1.12^{60}-1.12^{46}}{\log(1.12)}} = 0.6900072247$

If this happens, then his final average salary is $45000\left(\frac{1.04^{17}+1.04^{18}+1.04^{19}}{3}\right) = 91208.4928$

He has probability 0.3 of retiring at age 60, in which case the expected value of the accrued pension is $91208.4928 \times 13 \times 0.015\ddot{a}_{60}^{(12)} = 272954.8931$. The EPV of pension benefits from retirements at age 60 is therefore $272954.8931 \times 0.3 \times 0.6900072247(1.06)^{-14} = 24991.00164$.

To simplify, we assume the remaining retirements, except at age 65 happen in the middle of their year. We get the following:

age	P(retire)	$\ddot{a}_x^{(12)}$	S_{Fin}	EPV(Pension Benefits)
60.5	0.007984369979	15.22710298	93014.77692	947.3478018
61.5	0.007517395163	14.98002832	96735.3680	860.9124779
62.5	0.007113475961	14.72279172	100604.7827	785.5584389
63.5	0.006768439094	14.45514241	104628.9740	720.0204573
64.5	0.006478882979	14.17684657	108814.1330	663.1949110
65	0.4471424941	14.03363874	110969.0775	44878.8153482
total				48855.84944

So the total EPV of accrued pension benefits is $48855.84944 + 26792.25309 + 24991.00164 = \$100,639.10$

10.7 Funding the Benefits

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(a) Under the projected unit method, the final average salary is expected to be $47000\left(\frac{1.05^{18}+1.05^{19}+1.05^{20}}{3}\right) = 118860.9184$

We have $\ddot{a}_{65}^{(12)} = 13.95103541$. Therefore the EPV of accrued benefits for an individual who reaches retirement age is $118860.9184 \times 0.02 \times 26 \times 13.95103541(1.04)^{-20} = 393533.8372$.

The probability that this individual reaches retirement age is $_{20}p_{45} = e^{-0.0000076\left(\frac{1.1087^{65}-1.1087^{45}}{\log(1.1087)}\right)} = 0.948743622$. So the EPV of benefits is $0.948743622 \times 393533.8372 = 373362.7181$.

After another year, the projected final average salary will still be 113200.8747, so the EPV conditional on surviving to retirement age will be $118860.9184 \times 0.02 \times 27 \times 13.95103541(1.04)^{-19} = 425016.5442$.

The probability of surviving to retirement age is $e^{-0.0000076\left(\frac{1.1087^{65}-1.1087^{46}}{\log(1.1087)}\right)} = 0.9495331634$. The EPV of benefits at the end of the year is therefore $0.9495331634 \times 425016.5442 = 403567.3037$.

The accumulated value of the reserves at the begining of the year is $373362.7181(1.04) = 388297.2268$, so the annual contribution is $403567.3037 - 388297.2268 = \$15,270.08$.

(b) Under the traditional unit method, the final average salary is $47000\left(\frac{1.05^{-2}+1.05^{-1}+1}{3}\right) = 44797.43008$

The value in the current year is therefore $44797.43008 \times 0.02 \times 26 \times 13.95103541(1.04)^{-20}{}_{20}p_{45} = 140716.4818$

If the member survives the year, the final average salary in one year's time is $47000\left(\frac{1.05^{-1}+1+1.05}{3}\right) = 47037.30159$ so EPV at the end of next year is $47037.30159 \times 0.02 \times 27 \times 13.95103541(1.04)^{-20}{}_{20}p_{45} = 159705.2861$.

The accumulated value of the assets funding the benefit at the start of the year is $140716.4818(1.04) = 146345.1411$, so the contribution is $159705.2861 - 146345.1411 = \$13,360.14$.

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The rate of exit (for ages below 60) is $\mu_x^{01} + \mu_x^{02} = 0.2e^{-0.04x} + 0.00000187(1.130)^x$ so the probability of the employee remaining employed at age x is

$$e^{-\int_{46}^x 0.2e^{-0.04t} + 0.00000187(1.130)^t \, dt} = e^{-\left[\frac{0.00000187}{\log(1.130)}(1.130)^t - \frac{0.2}{0.04}e^{-0.04t}\right]_{46}^x} = e^{-0.004229764\left((1.130)^{x-46}-1\right) - 0.7940871\left(1-e^{-0.04(x-46)}\right)}$$

If the individual has retired, is at age x and has passed the guaranteed time of the pension, the value of the pension at that time is $R\sum_{i=0}^{\infty} {}_i p_x 1.05^{-i}$ where R is the regular pension payment. We have that

$${}_t p_x = e^{-0.00000187(1.13)^x \int_0^t e^{s\log(1.13)} \, ds} = e^{-0.00000187(1.13)^x \left(\frac{(1.13)^t-1}{\log(1.13)}\right)}$$

This gives that the value of the pension is

$$R\sum_{i=0}^{\infty} e^{-0.00000187(1.13)^x \left(\frac{(1.13)^i-1}{\log(1.13)}\right)} 1.05^{-i}$$

If the individual retires at age x, with pension value R then the present value of the pension at that time is

$$R\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x \left(\frac{(1.13)^i-1}{\log(1.13)}\right)} 1.05^{-i}\right)$$

and the present value is

$$(1.05)^{-19} R\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x \left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)} 1.05^{-i}\right)$$

If the member exits at age x, then the final average salary is $87000(1.06)^{x-46}\left(\frac{1+(1.06)^{-1}+(1.06)^{-2}}{3}\right) = 82168.39(1.06)^{x-46}$.

If the member withdraws at age x, then the final average salary is also $82168.39(1.06)^{x-46}$, and with COLA, the eventual accrued pension benefit is $82168.39(1.06)^{x-46}(1.02)^{65-x} = 20400.75(1.039216)^x$ per 100 years of service.

Since the employee has 12 years of service, the accrued pension beneifit if he withdraws at age x is

$$2448.090(1.039216)^x$$

The overall expected pension benefit for an individual who withdraws is therefore

$$\int_{46}^{60} 0.2e^{-0.04x} e^{-0.004229764\left((1.130)^{x-46}-1\right) - 0.7940871\left(1-e^{-0.04(x-46)}\right)} \times 2448.090(1.039216)^x \times$$

$$(1.05)^{-19} \left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x \left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)} 1.05^{-i}\right) \, dx$$

$$= 27393.66$$

After another year, the expected pension benefit to an individual who withdraws will be

$$\int_{47}^{60} 0.2e^{-0.04x}e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)} \times 2448.090(1.039216)^x \times$$

$$(1.05)^{-19}\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)}1.05^{-i}\right) dx$$

$$= 25057.95$$

In addition, annonther year will have accrued, so the expected benefit is $\frac{13}{12} \times 25057.95 = 27146.11$.
(Valued at the present time, not at age 47.)

If the individual is still employed at age 60, and retires at age x, then the present value of the accrued pension is

$$0.12 \times 87000(1 + 1.06^{-1} + 1.06^{-2})1.06^{x-46}1.05^{46-x}\left(\ddot{a}_{\overline{5}|0.05} + \sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^i-1}{\log(1.13)}\right)}1.05^{-i}\right)$$

Denote this value by $P(x)$. We then have that the expected pension payments to individuals who retire are

$$_{14}p_{46}^{(00)}\left(0.08P(60) + 0.92\int_0^5 0.1e^{-0.1t-0.00000187(1.13)^{60}\left(\frac{(1.13)^t-1}{\log(1.13)}\right)}P(60+t)\,dt + {}_{19}p_{46}P(65)\right) = 12053.66 {}_{14}p_{46}^{(00)}$$

We have that

$$_{14}p_{46}^{(00)} = e^{-0.004229764\left((1.130)^{14}-1\right)-0.7940871\left(1-e^{-0.04(14)}\right)} = 0.6979008$$

So the expected payments for individuals who retire are $0.6979008 \times 12053.66 = \$8412.26$.
The expected payments conditional on being employed at the end of the year are

$$12053.66 {}_{13}p_{47}^{(00)} = 12053.66 \times 0.7113312 = 8574.145$$

.

Multiplying by $\frac{13}{12}$ to account for the additional year of service, we get $\frac{13}{12} \times 8574.145 = 9288.66$.

Finally, conditional on the individual surviving for the year, the expected death benefits decrease by the expected death benefits for the year, which is

$$\int_0^1 0.00000187(1.13)^{46+t}e^{-\frac{0.00000187}{\log(1.13)}(1.13)^{46}(1.13^t-1)-5e^{-1.84}(1-e^{-0.04t})} \times 3 \times 87000(1.06^t)\,dt = 145.47$$

The increase in the present value of expected benefits is therefore

$$27146.11 - 27393.66 + 9288.66 - 8412.26 - 145.47 = \$483.38$$

The employee contribution is 4% of 87000, which is \$3,480.

11.2 The Yield Curve

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$\left(\frac{1.048^6}{1.043^3}\right)^{\frac{1}{3}} = \frac{1.048^2}{1.043} = 0.05302396932$, so the annual forward rate is 5.30%.

11.3 Valuation of Insurances and Life Annuities

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See slide. $\frac{300000 \times 0.001794442}{4.638626} = 116.05$.

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See slide.

11.4 Diversifiable and Non-diversifiable Risks

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(a) If the interest rate changes to 0.04 after 1 year, then the expected accumulated value of the premiums received at the end of the term is

$$91.95 \big((1.05)(1.04)^9 + 0.999977(1.04)^9 + 0.999952(1.04)^8 + 0.999926(1.04)^7 + 0.999898(1.04)^6 + 0.999868(1.04)^5$$
$$+ 0.999837(1.04)^4 + 0.999804(1.04)^3 + 0.999769(1.04)^2 + 0.999731(1.04)\big) = 1149.30$$

The expected accumulated value of the death benefits is

$$3200000 \big(0.000023(1.04)^9 + 0.000025(1.04)^8 + 0.000026(1.04)^7 + 0.000028(1.04)^6 + 0.000029(1.04)^5$$
$$+ 0.000031(1.04)^4 + 0.000033(1.04)^3 + 0.000035(1.04)^2 + 0.000038(1.04) + 0.000040\big) = 1160.443578$$

So the expected loss is $1160.44 - 1149.30 = \$11.15$

The present value of this loss is $11.15(1.04)^{-9}(1.05)^{-1} = 7.46$.

(b)

If the interest rate changes to 0.06 after 1 year, then the expected accumulated value of the premiums received at the end of the term is

$$91.625 \big((1.05)(1.06)^9 + 0.999977(1.06)^9 + 0.999952(1.06)^8 + 0.999926(1.06)^7 + 0.999898(1.06)^6 + 0.999868(1.06)^5$$
$$+ 0.999837(1.06)^4 + 0.999804(1.06)^3 + 0.999769(1.06)^2 + 0.999731(1.06)\big) = 1283.00$$

The expected accumulated value of the death benefits is

$$3200000 \big(0.000023(1.06)^9 + 0.000025(1.06)^8 + 0.000026(1.06)^7 + 0.000028(1.06)^6 + 0.000029(1.06)^5$$
$$+ 0.000031(1.06)^4 + 0.000033(1.06)^3 + 0.000035(1.06)^2 + 0.000038(1.06) + 0.000040\big) = 1261.99$$

So the expected profit is $1283.00 - 1261.99 = \$21.01$.

The present value of this profit is $21.01(1.06)^{-9}(1.05)^{-1} = 11.84$.

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If the interest rate in 1 years time is $i = 0.05$, then the variance of the present value of aggregate loss on N policies is

x	d_x	L	$L^2 d_x$
	0.000023	3047430.550	213597158
	0.000025	2902204.459	210569768
	0.000026	2763883.999	198615424
	0.000028	2632139.838	193988483
	0.000029	2506658.297	182216739
	0.000031	2387140.612	176651649
	0.000033	2273302.216	170540798
	0.000035	2164872.066	164033487
	0.000038	2061591.995	161506139
	0.000040	1963216.097	154168698
	0.999691	1064.587	1132995

Summing these gives the variance

$$\mathrm{Var}(L) = 1827021338$$

For the aggregate loss on N policies, we have

$$E(L) = -2.634162523N$$

$$\mathrm{Var}(L) = 1826854342N$$

Similarly, if interest rates fall to 1.04 in one year, PV future loss has

$$E(L) = 7.456161644N$$

$$\mathrm{Var}(L) = 1982791462N$$

If interest rates increase to 1.06 in one year:

$$E(L) = -11.84360483N$$

$$\mathrm{Var}(L) = 1690419800N$$

The overall variance of the aggregate loss is therefore

$$(0.2 \times 1690419800 + 0.6 \times 1826854342 + 0.2 \times 1982791462)N + 37.29465577N^2 = 1830855055N + 37.29465577N^2$$

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The premium is $200000(1.05)^{-1} \times 0.00032 = \$60.95$.

We have the following:

q_{58}	probability	$\mathbb{E}(L)$	$\mathrm{Var}(L)$
0.00028	0.26	-7.616666667	10155885.71
0.00032	0.58	0.00238095	11606262.13
0.00033	0.14	1.907142857	11968838.10
0.00077	0.02	85.71666667	27914996.83

So the variance of the aggregate loss for N policies is given by:

$0.26 \times 10155885.71 + 0.58 \times 11606262.13 + 0.14 \times 11968838.10 + 0.02 \times 27914996.83 = 11606099.59$

$1.907142857 \times 0.14 + 85.71666667 \times 0.02 - 7.616666667 \times 0.26 + 0.00238095 \times 0.58 = 0.00238095096$

$1.907142857^2 \times 0.14 + 85.71666667^2 \times 0.02 + 7.616666667^2 \times 0.26 + 0.00238095^2 \times 0.58 - 0.00238095096^2 = 162.5396825$

So the aggregate variance is $11606099.59N + 162.5396825N^2$.

11.5 Monte Carlo Simulation

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Probability of surviving t years is $e^{-\int_{47}^{47+t} 0.0000164(1.088)^x \, dx} = e^{-\frac{0.0000164(1.088)^{47}}{\log(1.088)}((1.088)^t - 1)} = e^{-0.01024147(1.088^t - 1)}$, so for the simulations

$$e^{-0.01024147(1.088^t - 1)} = u$$

$$0.01024147(1.088^t - 1) = -\log(u)$$

$$1.088^t = 1 - \frac{\log(u)}{0.01024147}$$

$$t = \frac{\log\left(1 - \frac{\log(u)}{0.01024147}\right)}{\log(1.088)}$$

For the interest rate, we have that $\log(i) - \log(0.04)$ is normally distributed with mean 0 and standard deviation 0.4.

$$i = 0.04e^{0.4\Phi^{-1}(v)}$$

We therefore have the following:

No.	u	t	v	i	Premiums	Benefits	PVFL
1	0.6129116	53.8	0.4397	0.03764344	199912.78	256717.52	56804.74
2	0.6120158	53.8	0.2549	0.03073115	199912.78	281238.80	81326.02
3	0.9504287	67.4	0.8275	0.05835981	199912.78	218301.91	18389.13

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Standard deviation of simulation mean is $\frac{64278.49}{100} = 642.7849$. Using a normal approximation, a 95% confidence interval is 1.96 standard deviations to either side of the mean. That is

$$[176.4161 - 642.7849 \times 1.96, 176.4161 + 642.7849 \times 1.96] = [-1083.44, 1436.27]$$

12.3 Profit Testing a Term Insurance Policy

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See slide.

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See slide.

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See slide.

12.5 Profit Measures

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See slide.

12.6 Using the Profit Test to Calculate the Premium

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At a risk discount rate of 10%, the NPV is

$$P\left((1.04)(1.1)^{-1} + 0.9996\left(\frac{1.1^{-1} - 1.1^{-10}}{0.1}\right)\right) - (160 + 56.34(1.1)^{-1} + 59.24(1.1)^{-2} + 62.50(1.1)^{-3} + 66.13(1.1)^{-4}$$

$$+ 69.93(1.1)^{-5} + 74.11(1.1)^{-6} + 78.65(1.1)^{-7} + 83.38(1.1)^{-8} + 88.83(1.1)^{-9} + 94.47(1.1)^{-10}) = 6.178837P - 590.908$$

The premium should be chosen to make this equal to zero. That is

$$6.178837P - 590.908 = 0$$
$$6.178837P = 590.908$$
$$P = \$95.63$$

12.7 Using the Profit Test to Calculate Reserves

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Recalling Question 45, with a premium of \$90, The net outflows without reserves are all positive except for the final year, where the net outflow is -4.61. To correct for this, the reserve R must satisfy $1.04R = 4.61$. This means $R = \$4.43$. This reserve is a negative cash flow at the end of year 9, so the net cash flow from year 9 is $1.03 - 4.43 = -3.40$. To prevent this negative cash flow, the reserve needs to satisfy $1.04R = 3.40$, so $R = 3.27$. This is a negative cash flow at the end of year 8, so the net cash-flow in that year is $6.48 - 3.27 = 3.21$, which is positive, so reserves of \$3.27 in year 9 and \$4.43 in year 10 are needed.

12.8 Profit Testing for Multiple-State Models

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Recall from Question 51 that the probabilities of the life being in each state are:

t	$_tp_{37}^{00}$	$_tp_{37}^{01}$	$_tp_{37}^{02}$
0	1	0	0
1	0.99812	0.000375	0.001505
2	0.99617	0.000750	0.003083
3	0.99414	0.001127	0.004736
4	0.99203	0.001505	0.006464
5	0.98985	0.001884	0.008271
6	0.98758	0.002263	0.010156
7	0.98523	0.002644	0.012123
8	0.98280	0.003025	0.014171
9	0.98029	0.003407	0.016303
10	0.97769	0.003790	0.018519

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If the life is in the healthy state at the start of year i, and is then aged x, the probability of being sick at the end is

$$e^{0.000001 \begin{pmatrix} -\left(1881 + \frac{1}{3} + 77x + x^2\right) & 375 + 2x & 1506 + \frac{1}{3} + 75x + x^2 \\ 67.5 + x & -(342.5 + 3x) & 275 + 2x \\ 0 & 0 & 0 \end{pmatrix}}$$

Probabilities of being sick or dead at the end of each year for a life alive at start of year are

t	p^{00}_{37+t}	p^{01}_{37+t}	p^{02}_{37+t}	p^{10}_{37+t}	p^{11}_{37+t}	p^{12}_{37+t}
0	0.9981204	0.0003745833	0.001504969	0.00006742499	0.9996576	0.0002750037
1	0.9980426	0.0003765658	0.001580836	0.00006842111	0.9996546	0.0002770063
2	0.9979628	0.0003785478	0.001658694	0.00006941708	0.9996516	0.0002790090
3	0.9978809	0.0003805293	0.001738542	0.00007041290	0.9996486	0.0002810119
4	0.9977971	0.0003825102	0.001820380	0.00007140856	0.9996456	0.0002830149
5	0.9977113	0.0003844905	0.001904206	0.00007240405	0.9996426	0.0002850181
6	0.9976235	0.0003864703	0.001990020	0.00007339939	0.9996396	0.0002870214
7	0.9975337	0.0003884495	0.002077823	0.00007439457	0.9996366	0.0002890249
8	0.9974420	0.0003904281	0.002167613	0.00007538957	0.9996336	0.0002910285
9	0.9973482	0.0003924062	0.002259390	0.00007638440	0.9996306	0.0002930323

t	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Net Cash Flow
0		200				-200
1	489.45		34.26	29.96666	300.9938	192.75108
2	489.45		34.26	30.12527	316.1673	177.41896
3	489.45		34.26	30.28383	331.7389	161.68881
4	489.45		34.26	30.44234	347.7084	145.56071
5	489.45		34.26	30.60081	364.0759	129.03478
6	489.45		34.26	30.75924	380.8412	112.11110
7	489.45		34.26	30.91762	398.0041	94.78978
8	489.45		34.26	31.07596	415.5646	77.07092
9	489.45		34.26	31.23425	433.5226	58.95464
10	489.45		34.26	31.39249	451.8780	40.44103

t	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Net Cash Flow
0		200				-200
1	489.45		34.26	79972.61	55.00074	-79503.89
2	489.45		34.26	79972.37	55.40126	-79504.06
3	489.45		34.26	79972.13	55.80181	-79504.22
4	489.45		34.26	79971.89	56.20238	-79504.38
5	489.45		34.26	79971.65	56.60299	-79504.54
6	489.45		34.26	79971.41	57.00362	-79504.70
7	489.45		34.26	79971.17	57.40429	-79504.86
8	489.45		34.26	79970.93	57.80498	-79505.02
9	489.45		34.26	79970.69	58.20571	-79505.18
10	489.45		34.26	79970.45	58.60646	-79505.34

We see that expected cash flows are all negative if the life starts the year in the sick state, and positive if the life starts the year in the healthy state. We need to calculate separate reserves for each state in the usual way by working backwards.

For the 10th year, if the life is in the sick state, the expected net cash flow is $-\$79,505.34$, so the reserve needed is $79505.34(1.07)^{-1} = 74304.06$.

For a life in the sick state at the start of year 9, the probability that the life is in the sick state at the end of year 9 is 0.9996306, so the expected reserves needed are $0.9996306 \times 74304.06 = \$74276.61$. For a life in the healthy state at the start of year 9, the probability of being in the sick state at the end is 0.0003924, so the additional expected cashflow is $-0.0003924 \times 74304.06 = -\$29.15691$, making the total expected net cash flow for that year $\$29.80$.

We proceed up the table in this way.

t	Reserve	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Expected Reserve	Net Cash Flow
2	530748.32	489.45		34.26	79972.37	55.40126	488396.64	-567900.70
3	488566.86	489.45		34.26	79972.13	55.80181	443262.32	-522766.54
4	443418.14	489.45		34.26	79971.89	56.20238	394953.03	-474457.41
5	395093.05	489.45		34.26	79971.65	56.60299	343245.02	-422749.56
6	343367.74	489.45		34.26	79971.41	57.00362	269064.28	-348568.98
7	269161.28	489.45		34.26	79971.17	57.40429	208497.72	-288002.58
8	208573.51	489.45		34.26	79970.93	57.80498	143668.64	-223173.66
9	143721.30	489.45		34.26	79970.69	58.20571	74276.61	-153781.79
10	74276.61	489.45		5235.55	79970.45	58.60646	0	-79505.34

There is no need to calculate the reserves for the first year, since we know that the life is in the healthy state at the start of year 1.

Knowing the reserves needed if the life is in the sick state, we can calculate the reserves as extra expenses in the healthy state:

t	Reserve	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Expected Reserve Sick	Expected Reserve Healthy	Net Cash Flow
0			200					21.65	-221.65
1	21.65	489.45		34.26	29.96666	300.9938	198.81	17.11	-23.17
2	17.14	489.45		34.26	30.12527	316.1673	183.98	11.78	-18.34
3	11.80	489.45		34.26	30.28383	331.7389	167.85	6.47	-12.63
4	6.49	489.45		34.26	30.44234	347.7084	150.34	2.15	-6.94
5	2.16	489.45		34.26	30.60081	364.0759	131.34		-2.31
6		489.45		34.26	30.75924	380.8412	103.49		8.62
7		489.45		34.26	30.91762	398.0041	80.61		14.18
8		489.45		34.26	31.07596	415.5646	55.83		21.24
9		489.45		34.26	31.23425	433.5226	29.00		29.95
10		489.45		34.26	31.39249	451.8780			40.44

The resulting profit signature is then

$$-221.65 \times 1 = -221.65$$
$$0 \times 1 = 0$$
$$0 \times 0.99812 = 0$$
$$0 \times 0.99617 = 0$$
$$0 \times 0.99414 = 0$$
$$0 \times 0.99203 = 0$$
$$8.62 \times 0.98985 = 8.53$$
$$14.18 \times 0.98758 = 14.00$$
$$21.24 \times 0.98523 = 20.93$$
$$29.95 \times 0.98280 = 29.43$$
$$40.44 \times 0.98029 = 39.64$$

13.4 Universal Life Insurance

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See slide.

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Profit signature is given by

Year	Profit vector	Probability in force	Profit signature
1	257.6443	1.0000000	257.6443
2	251.8084	0.9489626	238.9567
3	276.0940	0.9004360	248.6050
4	349.0236	0.8812669	307.5830
5	423.4032	0.8623908	365.1390
6	501.1323	0.8437907	422.8508
7	583.2349	0.8254506	481.4316
8	654.6027	0.7991134	523.1018
9	694.4401	0.7734466	537.1123
10	700.2219	0.7484175	524.0583
11	704.9742	0.7239944	510.3974
12	708.5417	0.7001450	496.0820
13	710.7165	0.6768365	481.0389
14	711.2783	0.6472935	460.4058
15	709.9592	0.6187523	439.2889

At the risk discount rate the NPV is

$$257.64(1.1)^{-1} + 238.96(1.1)^{-2} + 248.61(1.1)^{-3} + 307.58(1.1)^{-4} + 365.14(1.1)^{-5} + 422.85(1.1)^{-6} + 481.43(1.1)^{-7}$$
$$+523.10(1.1)^{-8} + 537.11(1.1)^{-9} + 524.06(1.1)^{-10} + 510.40(1.1)^{-11} + 496.08(1.1)^{-12} + 481.04(1.1)^{-13}$$
$$+460.41(1.1)^{-14} + 439.29(1.1)^{-15} - 1800 = 1117.60$$

The EPV of the premiums received is

$$3160(1 + 0.9489626(1.1)^{-1} + 0.9004360(1.1)^{-2} + 0.8812669(1.1)^{-3} + 0.8623908(1.1)^{-4} + 0.8437907(1.1)^{-5}$$
$$+0.8254506(1.1)^{-6} + 0.7991134(1.1)^{-7}) = 16615.07$$

The profit margin is $\frac{1117.60}{16615.07} = 0.067$

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We have

$$CoI_t = q_t(1.06)^{-1}(400000 - ((AV_{t-1} + 5160) \times 0.99 - CoI_t) * 1.05)$$
$$\left(1 - \frac{1.05}{1.06}q_t\right) CoI_t = q_t(1.06)^{-1}(400000 - (AV_{t-1} + 5160) \times 0.99)$$
$$CoI_t = \frac{q_t}{1.06 - 1.05q_t}(400000 - (AV_{t-1} + 5160) \times 0.99)$$

$$AV_t = 1.05\left(0.99(AV_{t-1} + 5160) - \frac{q_t}{1.06 - 1.05q_t}(400000 - (AV_{t-1} + 5160) * 0.99)\right)$$

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Profit signature is given by

Year	Profit vector	Probability in force	Profit signature
1	286.42	1.0000000	286.42
2	451.34	0.9489626	428.30
3	563.86	0.9004360	507.72
4	764.88	0.8812669	674.06
5	976.61	0.8623908	842.22
6	1193.12	0.8437907	1006.74
7	1420.48	0.8254506	1172.53
8	1660.71	0.7991134	1327.10
9	1907.88	0.7734466	1475.64
10	2161.98	0.7484175	1618.06
11	2423.01	0.7239944	1754.25
12	2690.95	0.7001450	1884.05

At the risk discount rate the NPV is

$$286.42(1.15)^{-1} + 428.30(1.15)^{-2} + 507.72(1.15)^{-3} + 674.06(1.15)^{-4} + 842.22(1.15)^{-5} + 1006.74(1.15)^{-6}$$
$$+1172.53(1.15)^{-7} + 1327.10(1.15)^{-8} + 1475.64(1.15)^{-9} + 1618.06(1.15)^{-10}$$
$$+1754.25(1.15)^{-11} + 1884.05(1.15)^{-12} - 2800 = 1769.39$$

The EPV of the premiums received is

$$5180(1 + 0.9489626(1.15)^{-1} + 0.9004360(1.15)^{-2} + 0.8812669(1.15)^{-3} + 0.8623908(1.15)^{-4} + 0.8437907(1.15)^{-5}$$
$$+0.8254506(1.15)^{-6} + 0.7991134(1.15)^{-7} + 0.7734466(1.15)^{-8} + 0.7484175(1.15)^{-9}$$
$$+0.7239944(1.15)^{-10} + 0.7001450(1.15)^{-11}) = 28233.07$$

The profit margin is $\frac{1769.39}{28233.07} = 0.063$

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After the premium is payed, the account value is \$123,389. The Expense charge is \$1,233.89, so after this charge, the account value is \$122,155.11. After applying the cost of insurance and interest, the account value is $\$130,705.97 - 1.07 CoI$. The death benefit is the larger of $2.3(130,705.97 - 1.07 CoI)$ and \$300,000. We first suppose that the death benefit is \$300,000, and calculate the cost of insurance charge. The Cost of insurance satisfies:

$$CoI = 0.000265(300000 - (130,705.97 - 1.07 CoI))(1.06)^{-1}$$
$$= 42.32 + 0.0002675 CoI$$
$$0.9997325 CoI = 42.32$$
$$CoI = \frac{42.32}{0.9997325} = 42.33$$

With this cost of insurance, the account value at the end of the year is $130,705.97 - 1.07 CoI = 130660.65$. This means that using the corridor factor requirement, the minimum death benefit is $2.3 \times 130660.65 =$

$300,519.50$, so the corridor factor requirement means that the death benefit is increased. We need to calculate the Cost of Insurance based on the new death benefit.

$$CoI = 0.000265(1.3(130,705.97 - 1.07CoI))(1.06)^{-1}$$
$$= 42.48 - 0.00034775CoI$$
$$1.00034775CoI = 42.48$$
$$CoI = \frac{42.48}{1.00034775} = 42.46$$

14.3 Deterministic Profit Testing for Equity-Linked Insurance

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See slide.

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See slide.

14.4 Stochastic Profit Testing

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See slide.

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See slides.

14.5 Stochastic Pricing

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See slide.

14.6 Stochastic Reserving

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(a) 1246.704

(b) 3610.32

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See slide.