ACSC/STAT 4720, Life Contingencies II
FALL 2015
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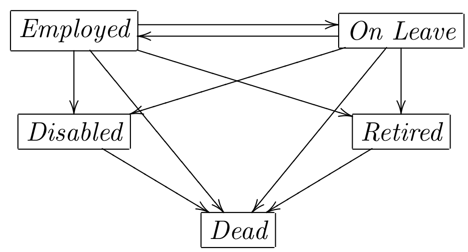
Sample Midterm Examination
Model Solutions

This Sample examination has more questions than the actual midterm, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *An insurance company is considering a new policy. The policy includes states with the following state diagram:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) Employed—Disabled—Retired—Dead

This is not possible, since it is impossible to transition from *Disabled* to *Retired*

(ii) Employed—On Leave—Retired—Dead

This is possible.

(iii) Employed—Retired—On Leave—Dead

This is not possible, since it is impossible to transition from *Retired* to *On Leave.*

(iv) Employed—On Leave—Employed—Retired—Dead

This is possible.

2. *Consider a permanent disability model with transition intensities*

$$\begin{aligned}
\mu_x^{01} &= 0.002 + 0.000005x \\
\mu_x^{02} &= 0.001 + 0.0000004x^2 \\
\mu_x^{12} &= 0.003 + 0.000004x
\end{aligned}$$

where State 0 is healthy, State 1 is permanently disabled and State 2 is dead. Write down an expression for the probability that an individual aged 29 is alive but permanently disabled at age 56. [You do not need to evaluate the expression, but should perform basic simplifications on it.]

The probability that the individual is in State 1 after 27 years is

$$
\begin{aligned}
{}_{27}p_{29}^{01} &= \int_0^{27} {}_tp_{29}^{\overline{00}}\mu_{29+t}^{01}\,{}_{27-t}p_{29+t}^{\overline{11}}\,dt \\
&= \int_0^{27} e^{-\int_0^t (0.003+0.000005(29+s)+0.0000004(29+s)^2)\,ds}(0.002+0.000005(29+t))e^{-\int_t^{27}(0.003+0.000004(29+s))\,ds}\,dt \\
&= \int_0^{27} e^{-\int_0^t (0.003+0.000145+0.000005s+0.00003364+0.0000232s+0.0000004s^2)\,ds}(0.002+0.000005(29+t))e^{-\int_t^{27}(0.003+} \\
&= \int_0^{27} e^{-\int_0^t (0.00317864+0.0000282s+0.0000004s^2)\,ds}(0.002+0.000005(29+t))e^{-\int_t^{27}(0.003116+0.000004s)\,ds}\,dt \\
&= \int_0^{27} e^{-(0.00317864t+0.0000141t^2+\frac{0.0000004}{3}t^3)\,ds}(0.002+0.000005(29+t))e^{-(0.003116(27-t)+0.000002(27^2-t^2))}\,dt \\
&= \int_0^{27} e^{-(0.00317864t+0.0000141t^2+\frac{0.0000004}{3}t^3)\,ds}(0.002+0.000005(29+t))e^{-(0.003116(27-t)+0.000002(27^2-t^2))}\,dt \\
&= \int_0^{27} e^{-(0.00317864t+0.0000141t^2+\frac{0.0000004}{3}t^3)\,ds}(0.002+0.000005(29+t))e^{-(0.08559-0.003116t-0.000002t^2)}\,dt \\
&= \int_0^{27} (0.002+0.000005(29+t))e^{-(0.08559+0.00006264t+0.0000121t^2+\frac{0.0000004}{3}t^3)\,ds}\,dt
\end{aligned}
$$

3. *A disability income model has transition intensities*

$$
\begin{aligned}
\mu_x^{01} &= 0.002 \\
\mu_x^{10} &= 0.001 \\
\mu_x^{02} &= 0.002 \\
\mu_x^{12} &= 0.004
\end{aligned}
$$

State 0 is healthy, State 1 is sick and State 2 is dead. Three actuaries calculate different values for the transition probabilities and benefit values. Which one has calculated plausible values? Justify your answer by explaining what is impossible about the values calculated by the other two actuaries.

Value	*Actuary I*	*Actuary II*	*Actuary III*
${}_2p_{37}^{(00)}$	*0.992036*	*0.992036*	*0.992036*
${}_2p_{37}^{(01)}$	*0.003960*	*0.003968*	*0.003964*
${}_4p_{37}^{(01)}$	*0.007857*	*0.007857*	*0.007857*
${}_4p_{37}^{(02)}$	*0.015857*	*0.008000*	*0.008000*
${}_4p_{37}^{(12)}$	*0.008000*	*0.015857*	*0.015857*
${}_2p_{39}^{(01)}$	*0.003960*	*0.003968*	*0.003964*
${}_2p_{39}^{(11)}$	*0.992054*	*0.992054*	*0.990054*

We have that ${}_4p_{37}^{(01)} = {}_2p_{37}^{(00)} \times {}_2p_{39}^{(01)} + {}_2p_{37}^{(01)} \times {}_2p_{39}^{(11)}$. For the numbers in the table, this gives

$$0.007857 = 0.992036 \times 0.003960 + 0.003960 \times 0.992054 = 0.007857$$
$$0.007857 = 0.992036 \times 0.003968 + 0.003968 \times 0.992054 = 0.007873$$
$$0.007857 = 0.992036 \times 0.003964 + 0.003964 \times 0.990054 = 0.007857$$

So the second actuary's calculations cannot be right. Furthermore, since $\mu_x^{02} < \mu_x^{12}$ for all x, we should have ${}_4p_{37}^{02} <_4 p_{37}^{12}$, which rules out the first actuary's calculations. This means that only Actuary III's calculations might be correct. [Indeed these are the correct values.]

4. *A disability income model has the following four states:*

State	*Meaning*
0	*Healthy*
1	*Sick*
2	*Accidental Death*
3	*Other Death*

The transition intensities are:

$$\begin{aligned}
\mu_x^{01} &= 0.001\\
\mu_x^{02} &= 0.002\\
\mu_x^{03} &= 0.001\\
\mu_x^{10} &= 0.002\\
\mu_x^{12} &= 0.001\\
\mu_x^{13} &= 0.003
\end{aligned}$$

You calculate that the probability that the life is healthy t years from the start of the policy is $0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t}$, and the probability that the life is sick t years from the start of the policy is $0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t}$.

Calculate the premium for a 5-year policy with premiums payable continuously while the life is in the healthy state, which pays no benefits while the life is in the sick state, but pays a benefit of \$200,000 in the event of accidental death and a benefit of \$100,000 in the event of other death. The interest rate is $\delta = 0.03$.

We calculate

$$\begin{aligned}
\overline{a}_{x:\overline{5}|} &= \int_0^5 e^{-0.03t}(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})\,dt\\
&= \int_0^5 0.2113249e^{-0.036732051t}\,dt + \int_0^5 0.7886751e^{-0.033267949t})\,dt\\
&= \frac{0.2113249}{0.036732051}(1 - e^{-0.036732051\times 5}) + \frac{0.7886751}{0.033267949}(1 - e^{-0.033267949\times 5})\\
&= 4.598130
\end{aligned}$$

The EPV of the benefits to lives who die accidentally from State 0 are given by

$$
\begin{aligned}
&200000 \int_0^5 0.002(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t}\,dt \\
&= 400 \int_0^5 (0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t}\,dt \\
&= 400 \times 4.598130 \\
&= 1839.25
\end{aligned}
$$

The EPV of the benefits to lives who die otherwise from State 0 are given by

$$
\begin{aligned}
&100000 \int_0^5 0.001(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t}\,dt \\
&= 100 \int_0^5 (0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})e^{-0.03t}\,dt \\
&= 100 \times 4.598130 \\
&= 459.81
\end{aligned}
$$

The EPV of the benefits to lives who die accidentally from State 1 are given by

$$
\begin{aligned}
&200000 \int_0^5 0.001(0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t})e^{-0.03t}\,dt \\
&= 200 \int_0^5 0.2886752e^{-0.033267949t} - 0.2886752e^{-0.036732051t})\,dt \\
&= \frac{57.73504}{0.033267949}(1 - e^{-5\times 0.033267949}) - \frac{57.73504}{0.036732051}(1 - e^{-5\times 0.036732051}) \\
&= 2.226627
\end{aligned}
$$

The EPV of the benefits to lives who die otherwise from State 1 are given by

$$
\begin{aligned}
&100000 \int_0^5 0.003(0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t})e^{-0.03t}\,dt \\
&= 300 \int_0^5 0.2886752e^{-0.033267949t} - 0.2886752e^{-0.036732051t})\,dt \\
&= 3.339940
\end{aligned}
$$

The total EPV of benefits is therefore $1839.25 + 459.81 + 2.23 + 3.34 = \$2,304.63$. The annual rate of premium is $\frac{2304.63}{4.598130} = \$501.21$.

5. *Under a certain model for transition intensities in a critical illness model, with the following transition diagram:*

$$\begin{tikzcd}
\boxed{\textit{0 Healthy}} \arrow[r] \arrow[dr] & \boxed{\textit{1 Critically Ill}} \arrow[d] \\
& \boxed{\textit{2 Dead}}
\end{tikzcd}$$

you calculate:

$$\begin{array}{lll}
{}_5p^{00}_{41} = 0.866102 & {}_5p^{01}_{41} = 0.0542667 & {}_5p^{02}_{41} = 0.0796309 \\
\overline{a}^{00}_{41} = 13.5501 & \overline{a}^{01}_{41} = 2.48302 & \overline{a}^{02}_{41} = 8.96688 \\
\overline{a}^{0,0}_{46} = 13.1355 & \overline{a}^{0,1}_{46} = 2.49464 & \overline{a}^{0,2}_{46} = 9.36984 \\
\overline{a}^{1,1}_{46} = 13.2984 & \overline{a}^{1,2}_{46} = 11.7016 & \\
\overline{A}^{01}_{41} = 0.196752 & \overline{A}^{02}_{41} = 0.358682 & \overline{A}^{01}_{46} = 0.202971 \\
\overline{A}^{02}_{46} = 0.374801 & \overline{A}^{12}_{46} = 0.468071 &
\end{array}$$

where 0 is healthy, 1 is critically ill, and 2 is dead. Calculate the premium for a 5-year policy for a life aged 41, with continuous premiums payable while in the healthy state, which pays a benefit $280,000 immediately upon death in the case of death directly from the healthy state and a benefit of $190,000 upon entry to the critically ill state, followed by a further benefit of $140,000 upon death after diagnosis of critical illness. [Hint: You need to separate the death benefits into two cases — cases where the life is critically ill first, and cases where the life is not critically ill first. You can calculate the value for cases where the life is not critically ill first by calculating the value of a payment upon first exit from State 0, which can be calculated from $a^{00}_{41:\overline{5}|}$.

We first calculate $a^{00}_{41:\overline{5}|} = a^{00}_{41} - 0.866102a^{00}_{46}e^{-0.2} = 13.5501 - 0.866102 \times 13.1355e^{-0.2} = 4.23566$

The EPV of the critical illness benefits is given by calculating $\overline{A}^{01}_{41:\overline{5}|} = \overline{A}^{01}_{41} -{}_5\,p^{00}_{41}e^{-0.2}\overline{A}^{01}_{46} = 0.196752 - 0.866102 \times 0.202971e^{-0.2} = 0.05282438$. The EPV of the critical illness benefits is $0.05282438 \times 190000 = 10036.63$.

The EPV of death benefits is harder to calculate, since the death benefits are reduced if the individual becomes critically ill before death. We have that $\overline{A}^{02}_{41:\overline{5}} = \overline{A}^{02}_{41} - 0.866102e^{-0.2}\overline{A}^{02}_{46} - 0.0542667e^{-0.2}\overline{A}^{12}_{46} = 0.358682 - 0.866102e^{-0.2}0.374801 - 0.0542667e^{-0.2}0.468071 = 0.1137053$.

We can solve the problem of what proportion of these are critically ill first by considering the value of a payment immediately upon any exit from the healthy state. This is given by $1-\delta a^{00}_{41} = 1-0.04\times 13.5501 = 1-0.542004 = 0.457996$. Subtracting the value of payments for entry to the critically ill state gives the EPV of payments for death directly from the healthy state as $0.457996-0.196752 = 0.261244$ and for payments for death directly from the healthy state for individuals aged 46, we get $1 - \delta a^{00}_{46} = 1 - 0.04 \times 13.1355 = 1 - 0.52542 = 0.47458$, and $0.47458 - 0.202971 = 0.271609$. Total payments from direct death are therefore given by $0.261244 - 0.866102e^{-0.2} \times 0.271609 = 0.06864488$

Total payments from deaths that are critically ill first are therefore $0.1137053 - 0.06864488 = 0.04506041$.

The total EPV of death benefits is therefore $0.06864488 \times 280000 + 0.04506041 \times 140000 = 25529.02$.

The total EPV of all benefits is therefore, $25529.02 + 10036.63 = \$35,565.65$, so the premium is $\frac{35565.65}{} = \$8,396.72$.

6. *The following is a multiple decrement table giving probabilities of surrender (decrement 1) and death (decrement 2) for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
49	10000.00	235.54	1.46
50	9763.00	222.44	1.55
51	9539.01	210.28	1.65
52	9327.08	198.99	1.77

A life insurance policy has a death benefit of \$400,000 payable at the end of the year of death. Premiums are payable at the beginning of each year. Calculate the premium for a 4-year annual policy sold to a life aged 49 if there is no-payment to policyholders who surrender their policy, and the interest rate is $i = 0.06$.

The expected death benefit is $A^{02}_{49:\overline{4}|} = 0.000146(1.06)^{-1} + 0.000155(1.06)^{-2} + 0.000165(1.06)^{-3} + 0.000177(1.06)^{-4} = 0.0005544231$, so $400000 \times 0.0005544231 = \$221.77$. $a^{00}_{49:\overline{4}|} = 1 + 0.9763(1.06)^{-1} + 0.953901(1.06)^{-2} + 0.932708(1.06)^{-3} = 3.553126$, so the annual premium is $\frac{221.77}{3.553126} = \$62.42$.

7. *Update the multiple decrement table below*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
58	10000.00	176.04	2.68
59	9823.96	167.67	2.88
60	9656.29	159.84	3.10
61	9496.46	152.50	3.34
62	9343.96	145.62	3.60
63	9198.34	139.16	3.89

with the following mortality probabilities

x	l_x	d_x
58	10000.00	1.81
59	9998.19	1.92
60	9996.27	2.04
61	9994.22	2.18
62	9992.05	2.32
63	9989.73	2.47

[The first decrement is surrender, the second is death.] Using:

(a) UDD in the multiple decrement table.

Under UDD in the multiple decrement table, the relation between single decrements and multiple decrements is.

$$1 - q^{1*} = (1 - q^{01} - q^{02})^{\frac{q^{01}}{q^{01}+q^{02}}}$$

$$1 - q^{2*} = (1 - q^{01} - q^{02})^{\frac{q^{02}}{q^{01}+q^{02}}}$$

We therefore find that the individual decrement probabilities for surrender are

x	q_x^{*1}
58	0.01760637
59	0.01707463
60	0.01656515
61	0.01607614
62	0.01560748
63	0.01515776

Now we update the table with these surrender probabilities and the new death probabilities. The new multiple decrement probabilities are the solution to

$$\begin{aligned} 1 - q^{1*} &= (1 - q^{01} - q^{02})^{\frac{q^{01}}{q^{01}+q^{02}}} \\ 1 - q^{2*} &= (1 - q^{01} - q^{02})^{\frac{q^{02}}{q^{01}+q^{02}}} \end{aligned}$$

Which are given by

$$\begin{aligned} 1 - q^{01} - q^{02} &= (1 - q^{*1})(1 - q^{*2}) \\ \frac{q^{01}}{q^{02}} &= \frac{\log(1 - q^{*1})}{\log(1 - q^{*2})} \end{aligned}$$

The new multiple decrement table is therefore

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
10000.00	176.05	1.79	
9822.16	167.69	1.87	
9652.59	159.88	1.95	
9490.76	152.56	2.05	
9336.15	145.70	2.15	
9188.30	139.26	2.25	

(b) UDD in the independent decrements.

For UDD in the individual decrement tables, the decrement probabilities are related by:

$$\begin{aligned} q^{01} + q^{02} &= 1 - (1 - q^{*1})(1 - q^{*2}) \\ \frac{q^{01}}{q^{02}} &= \frac{q^{*1}}{q^{*2}} \\ \left(1 - \frac{q^{02}}{q^{01}} q^{*1}\right)(1 - q^{*1}) &= 1 - q^{01} - q^{02} \\ \frac{q^{02}}{q^{01}} (q^{*1})^2 - \left(1 + \frac{q^{02}}{q^{01}}\right) q^{*1} + q^{01} + q^{02} &= 0 \\ q^{*1} &= \frac{q^{01} + q^{02} - \sqrt{(q^{01} + q^{02})^2 - 4q^{01}q^{02}(q^{01} + q^{02})}}{2q^{02}} \end{aligned}$$

We therefore find that the individual decrement probabilities for surrender are

x	q_x^{*1}
58	0.01760865
59	0.01707704
60	0.01656770
61	0.01607883
62	0.01561033
63	0.01516079

Now we update the table with these surrender probabilities and the new death probabilities. The new multiple decrement probabilities are given by

$$\begin{aligned} q^{01} + q^{02} &= 1 - (1 - q^{*1})(1 - q^{*2}) \\ \frac{q^{01}}{q^{02}} &= \frac{q^{*1}}{q^{*2}} \\ q^{02} &= \frac{1 - (1 - q^{*1})(1 - q^{*2})}{\left(1 + \frac{q^{*1}}{q^{*2}}\right)} \\ &= q^{*2}\left(\frac{q^{*1} + q^{*2} - q^{*1}q^{*2}}{q^{*2} + q^{*1}}\right) \end{aligned}$$

The new multiple decrement table is therefore

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
10000.00	176.05	1.81	
9822.14	167.70	1.89	
9652.55	159.89	1.97	
9490.69	152.57	2.07	
9336.05	145.71	2.17	
9188.18	139.27	2.27	

8. *The mortalities for a husband and wife (whose lives are assumed to be independent) aged 62 and 53 respectively, are given in the following tables:*

x	l_x	d_x
62	*10000.00*	*5.31*
63	*9994.69*	*5.76*
64	*9988.93*	*6.25*
65	*9982.68*	*6.79*
66	*9975.89*	*7.37*
67	*9968.52*	*8.01*

x	l_x	d_x
53	*10000.00*	*3.03*
54	*9996.97*	*3.25*
55	*9993.72*	*3.48*
56	*9990.24*	*3.74*
57	*9986.49*	*4.03*
58	*9982.47*	*4.33*

The interest rate is $i = 0.03$.

(a) They want to purchase a 5-year joint life insurance policy with a death benefit of \$2,500,000. Annual premiums are payable while both are alive. Calculate the net premium for this policy using the equivalence principle.

The EPV of the premium is given by $\ddot{a}_{62,53:\overline{5}|} = 1 + 0.999469 \times 0.999697(1.03)^{-1} + 0.998893 \times 0.999372(1.03)^{-2} + 0.998268 \times 0.999024(1.03)^{-3} + 0.997589 \times 0.998649(1.03)^{-4} = 4.708838$

We also calculate

$A_{62,53:\overline{5}|} = (1-0.999469\times0.999697)(1.03)^{-1}+(0.999469\times0.999697-0.998893\times0.999372)(1.03)^{-2}+(0.998893 \times 0.999372 - 0.998268 \times 0.999024)(1.03)^{-3} + (0.998268 \times 0.999024 - 0.997589 \times 0.998649)(1.03)^{-4} + (0.997589 \times 0.998649 - 0.996852 \times 0.998247)(1.03)^{-5} + = 0.004463484$

So the EPV of the benefits is $2500000 \times 0.004463484 = \$11,158.71$, so the premium is

$$\frac{11158.71}{4.708838} = \$2,369.74$$

(b) They want to purchase a 5-year reversionary annuity, which will provide an annuity to the husband of \$60,000 at the end of each year for the 5-year term if the wife is dead and the husband is alive. Calculate the net annual premium for this policy using the equivalence principle.

The probability that the wife is dead and the husband is alive at the end of each year is given in the following table:

Year	Probability	EPV of payment
1	$(1 - 0.999697) \times 0.999469 = 0.000303$	$0.000303(1.03)^{-1} = 0.000294$
2	$(1 - 0.999372) \times 0.998893 = 0.000627$	$0.000627(1.03)^{-2} = 0.000591$
3	$(1 - 0.999024) \times 0.998268 = 0.000974$	$0.000974(1.03)^{-3} = 0.000892$
4	$(1 - 0.998650) \times 0.997589 = 0.001347$	$0.001347(1.03)^{-4} = 0.001197$
5	$(1 - 0.998247) \times 0.996852 = 0.001747$	$0.001747(1.03)^{-5} = 0.001507$

Summing the last column gives $a^{01}_{63,52:\overline{5}|} = 0.004480903$, so the EPV of benefits is $0.004480903 \times 60000 = \$268.85$, and the premium is $\frac{268.85}{4.708838} = \$57.10$.

(c) They want to purchase a 5-year last survivor insurance policy, with a death benefit of \$120,000,000. Premiums are payable while either life is alive. Calculate the net premium for this policy using the equivalence principle.

Year	Probability Both Dead	Probability of Payment	EPV of payment	EPV of
1	$(1 - 0.999469) \times (1 - 0.999697) = 0.000000161$	0.000000161	0.000000156	0.970873
2	$(1 - 0.998893) \times (1 - 0.999372) = 0.000000695$	0.000000534	0.000000504	0.942595
3	$(1 - 0.998268) \times (1 - 0.999024) = 0.000001690$	0.000000995	0.000000911	0.915140
4	$(1 - 0.997589) \times (1 - 0.998649) = 0.000003257$	0.000001567	0.000001390	0.888484
5	$(1 - 0.996852) \times (1 - 0.998247) = 0.000005518$	0.000002261	0.000001953	0.862604

This gives $\ddot{a}_{\overline{62,53:5}} = 4.717093$ and $A_{\overline{62,53:5}} = 0.000004913182$, so the premium is $\frac{0.000004913182 \times 120000000}{4.717093} = \$124.99$.

9. *A husband is 64; the wife is 73. Their lifetables while both are alive, and the lifetable for the husband if the wife is dead, are given below:*

x	l_x	d_x
64	*10000.00*	*6.92*
65	*9993.08*	*7.49*
66	*9985.59*	*8.12*
67	*9977.48*	*8.80*
68	*9968.68*	*9.55*
69	*9959.13*	*10.36*

x	l_x	d_x
73	*10000.00*	*31.73*
74	*9968.27*	*34.69*
75	*9933.58*	*37.92*
76	*9895.66*	*41.45*
77	*9854.20*	*45.30*
78	*9808.91*	*49.49*

x	l_x	d_x
64	*10000.00*	*11.56*
65	*9988.44*	*12.56*
66	*9975.88*	*13.65*
67	*9962.23*	*14.83*
68	*9947.40*	*16.12*
69	*9931.28*	*17.53*

Calculate the probability that the husband survives to the end of the 5-year period. Use the UDD assumption for handling changes to the husband's mortality in the event of the wife's death.

In the year of the wife's death, let q_a be the probability that the husband dies if the wife is alive for the whole year, and let q_d be the probability that the husband dies if the wife is dead for the whole year. If the wife's death occurs at time t, then the husband's probability of surviving the year is $(1 - tq_a)\frac{1-q_d}{1-tq_d}$. The overall probability of the husband surviving the year if the wife's death is uniformly distributed is therefore

$$\begin{aligned}\int_0^1 (1-q_d)\frac{1-tq_a}{1-tq_d}\,dt &= (1-q_d)\int_0^1 \left(\frac{q_a}{q_d} - \left(\frac{q_a-q_d}{{q_d}^2}\right)\frac{q_d}{1-tq_d}\right)dt \\ &= (1-q_d)\left(\frac{q_a}{q_d} - \left(\frac{q_a-q_d}{{q_d}^2}\right)\log(1-tq_d)\right)\end{aligned}$$

This gives the following table:

Year	P(W Dies)	P(H survives to start of year)	P(H survives year)	P(H survives from end of year)	Total P(H survives 5 years)
1	0.003173	1.000000	0.9990759	$\frac{9931.28}{9988.44}$	0.003151927
2	0.003469	0.999308	0.9989964	$\frac{9931.28}{9975.88}$	0.003447638
3	0.003792	0.998559	0.9989091	$\frac{9931.28}{9962.23}$	0.003770620
4	0.004145	0.997748	0.9988145	$\frac{9931.28}{9947.40}$	0.004124069
5	0.004530	0.996868	0.9987106	$\frac{9931.28}{9931.28}$	0.004509989
>5	0.980891	0.995913	1	1	0.976882098

The total probability that the husband survives the 5 years is therefore $0.003151927 + 0.003447638 + 0.003770620 + 0.004124069 + 0.004509989 + 0.976882098 = 0.9958863$.

10. *A couple want to receive the following:*

- *While both are alive, they would like to receive a pension of \$90,000 per year.*
- *If the husband is alive and the wife is not, they would like to receive a pension of \$85,000 per year.*
- *If the wife is alive and the husband is not, they would like to receive a pension of \$65,000 per year.*
- *When one dies, if the husband dies first, they would like to receive \$92,000, if the wife dies first, they would like to receive \$120,000.*
- *When the second one dies, if it is the husband, they would like to receive a benefit of \$65,000; if it is the wife, they would like to receive a benefit of \$93,000.*

Construct a combination of insurance and annuity policies that achieve this combination of benefits.

There are many possible solutions. Below are two of them.

First solution:

- A last survivor annuity for \$65,000.
- A life annuity for the husband for \$20,000.
- A joint life annuity for \$5,000.
- A life insurance policy for \$65,000 for the husband.

- A life insurance policy for $93,000 for the wife.
- A joint life insurance policy for $27,000.

This gives the following:

- While both are alive, an annuity of $5000 + 65000 + 20000 = 90,000$.
- While the wife is dead and the husband is alive, an annuity of $65000 + 20000 = 85,000$.
- While the wife is alive and the husband is dead, an annuity of $\$65,000$.
- If the wife dies first, a death benefit of $93000 + 27000 = \$120,000$.
- If the husband dies first, a death benefit of $65000 + 27000 = \$92,000$.
- If the wife dies second, a death benefit of $\$93,000$.
- If the husband dies second, a death benefit of $\$65,000$.

Second solution:

- A life annuity for the wife for $5,000.
- A life annuity for the husband for $85,000.
- A reversionary annuity for $60,000 while the husband is dead and the wife is alive.
- A last survivor insurance policy for $65,000.
- A life insurance policy for $28,000 for the wife.
- A joint life insurance policy for $92,000.

This gives the following:

- While both are alive, an annuity of $85000 + 5000 = \$90,000$.
- While the wife is dead and the husband is alive, an annuity of $\$85,000$.
- While the wife is alive and the husband is dead, an annuity of $\$65,000$.
- If the wife dies first, a death benefit of $92000 + 28000 = \$120,000$.
- If the husband dies first, a death benefit of $\$92,000$.
- If the wife dies second, a death benefit of $65000 + 28000 = \$93,000$.
- If the husband dies second, a death benefit of $\$65,000$.

11. *A husband aged 52 and wife aged 66 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.000003y + 0.0000001x \\
\mu_{xy}^{02} &= 0.0000015x + 0.0000004y \\
\mu_{xy}^{03} &= 0.000042 + 0.000013x + 0.000019y \\
\mu_{x}^{13} &= 0.000004x \\
\mu_{x}^{23} &= 0.000003y
\end{aligned}$$

Which of the following expressions gives the probability that after 7 years, the husband is dead and the wife is alive? Justify your answer.

(i) $\int_0^7 e^{-(0.0015595+0.0020203t+0.0000205t^2)}(0.00003965+0.0000039t)\,dt$

(ii) $\int_0^7 e^{-(0.0023614+0.0014475t+0.0000205t^2)}(0.00003465+0.0000019t)\,dt$

(iii) $\int_0^7 e^{-(0.0015595+0.0019496t+0.0000170t^2)}(0.00003465+0.0000019t)\,dt$

(iv) $\int_0^7 e^{-(0.0009948+0.0020203t+0.0000150t^2)}(0.00003465+0.0000019t)\,dt$

The probability of this is

$$\begin{aligned}
\int_0^7 {}_tp_{52,66}^{00}\mu_{52+t,66+t}^{02}{}_{7-t}p_{52+t,66+t}^{22}\,dt &= \int_0^7 e^{-\int_0^t 0.0000224(66+s)+0.0000146(52+s)+0.000042\,ds}(0.00000015(55+t)+0.00 \\
&= \int_0^7 e^{-\int_0^t 0.0021476+0.0000370s\,ds}(0.00003465+0.0000019t)e^{-\int_t^7 0.000198+0.0000} \\
&= \int_0^7 e^{-(0.0021476t+0.0000185t^2)}(0.00003465+0.0000019t)e^{-(0.000198(7-t)+0.0000} \\
&= \int_0^7 e^{-(0.0015595+0.0019496t+0.0000170t^2)}(0.00003465+0.0000019t)\,dt
\end{aligned}$$

So the answer is (iii).

12. *An individual aged 42 has a current salary of \$76,000 for the coming year. The salary scale is $s_y = 1.05^y$. Estimate the individual's final average salary (average of last 3 years working) assuming the individual retires at exact age 65.*

    If the individual retires at age 65, the final average salary is $76000\frac{((1.05)^{20}+(1.05)^{21}+(1.05)^{22})}{3} = \$211,901.20$.

13. *An employer sets up a DC pension plan for its employees. The target replacement ratio is 60% of final average salary for an employee who enters the plan at exact age 30, with the following assumptions:*

    - *At age 65, the employee will purchase a continuous life annuity, plus a continuous reversionary annuity for the employee's spouse, valued at 60% of the life annuity.*
    - *At age 65, the employee is married to someone aged 63.*
    - *The salary scale is $s_y = 1.04^y$.*
    - *Mortalities are independent and given by $\mu_x = 0.0000016(1.092)^x$. The value of the life annuity is based on $\delta = 0.045$. This gives $\overline{a}_{65} = 19.63036$, $\overline{a}_{63} = 19.83656$ and $\overline{a}_{65,63} = 18.7867$.*
    - *A fixed percentage of salary is payable annually in arrear.*
    - *Contributions earn an annual rate of 7%.*

    Calculate the percentage of salary payable annually to achieve the target replacement rate under these assumptions.

    Suppose the employee has a current salary at age 30 for the coming year of 1. At retirement, the employee's final average salary is $\frac{1.04^{32}+1.04^{33}+1.04^{34}}{3} = 3.650252$. With a replacement ratio of 60%, this leads to an annuity of $0.6 \times 3.650252 = 2.1901512$ and a reversionary annuity of $0.6 \times 2.1901512 = 1.31409072$. For the reversionary annuity, $\overline{a}_{65|63} = \overline{a}_{63} -$

$\overline{a}_{65,63} = 19.83656 - 18.7867 = 1.04986$, so the total EPV of the annuities at age 65 is $2.1901512 \times 19.63036 + 1.31409072 \times 1.04986 = 44.37307$.

The contributions should therefore be chosen so that the accumulated value of contributions by age 65 is 44.37307. If the employee pays all salary into the pension plan, the accumulated value will be $\frac{1.07^{35}-1.04^{35}}{1.07-1.04} = 224.3497$. The percentage of salary that needs to be paid into the plan is therefore $\frac{44.37307}{224.3497} = 19.78\%$.

14. *The salary scale is given in the following table:*

y	s_y	y	s_y	y	s_y	y	s_y
30	1.000000	39	1.350398	48	1.845766	57	2.553877
31	1.033333	40	1.397268	49	1.912422	58	2.649694
32	1.067933	41	1.445983	50	1.981785	59	2.749515
33	1.103853	42	1.496620	51	2.053975	60	2.853522
34	1.141149	43	1.549263	52	2.129115	61	2.961903
35	1.179879	44	1.604000	53	2.207337	62	3.074855
36	1.220103	45	1.660921	54	2.288777	63	3.192585
37	1.261887	46	1.720122	55	2.373580	64	3.315310
38	1.305295	47	1.781702	56	2.461894	65	3.443256

An employee aged 42 and 4 months has 12 years of service, and a current salary of $106,000 (for the coming year). She has a defined benefit pension plan with $\alpha = 0.02$ and S_{Fin} is the average of her last 3 years' salary. The employee's mortality is given by $\mu_x = 0.00000195(1.102)^x$. The pension benefit is payable monthly in advance. The interest rate is $i = 0.05$. This results in $\ddot{a}_{65}^{(12)} = 17.15373$ and ${}_{22.66666667}p_{42.33333333} = 0.9901951$. There is no death benefit, and there are no exits other than death or retirement at age 65.

(a) Calculate the EPV of the accrued benefit using the projected unit method under the assumption that the employee retires at age 65. [Calculate the salary scale at non-integer ages by linear interpolation.]

We interpolate $s_{42.33333} = \frac{2}{3} \times 1.496620 + \frac{1}{3} \times 1.549263 = 1.514168$. The employee's final average salary is therefore $106000 \times \frac{3.192585+3.315310+3.443256}{1.514168\times3} = 232211.57$. The accrued benefit is a pension with annual payment rate $0.02\times12\times232211.57\times = \$55,730.78$. The EPV of this benefit when the employee reaches age 65 is therefore $55730.78\times17.15373 = 955990.75$. The current EPV is $955990.75 \times 0.9901951(1.05)^{-22.666667} = \$313,244.67$.

(b) Calculate the employer's contribution for this employee for the year. [${}_{21.66666667}p_{43.33333333} = 0.9903189$.]

In a year's time, the accrued benefit will be a pension with annual payment rate $0.02 \times 13 \times 232211.57\times = \$60,375.01$. The EPV of this is $60375.01\times17.15373\times0.9903189(1.05)^{-21.666666667} = 356360.35$. The accumulated value of the previous contributions is $313244.67 \times 1.05 = 328906.90$, so this year's contribution is $356360.35 - 328906.90 = \$27,453.45$.

15. *The service table is given below:*

x	l_x	1	2	3
40	10000.00	118.76	0	0.51
41	9880.73	112.29	0	0.58
42	9767.86	107.16	0	0.65
43	9660.05	101.84	0	0.73
44	9557.49	96.80	0	0.82
45	9459.86	92.02	0	0.93
46	9366.91	87.50	0	1.04
47	9278.37	83.19	0	1.18
48	9193.99	80.11	0	1.32
49	9112.57	75.21	0	1.49
50	9035.87	71.48	0	1.68
51	8962.71	67.92	0	1.89
52	8892.90	64.51	0	2.12
53	8826.26	61.23	0	2.39
54	8762.64	58.07	0	2.69
55	8701.88	55.03	0	3.03
56	8643.83	52.06	0	3.41
57	8588.36	49.18	0	3.84
58	8535.34	46.37	0	4.32
59	8484.64	43.62	0	4.86
60^-	8484.64		1098.84	
60	7385.80	21.70	819.91	5.79
61	6538.40	18.30	611.98	6.38
62	5901.74	10.81	384.29	5.86
63	5500.78	9.14	639.20	6.15
64	4846.29	7.73	351.32	6.10
65^-	4481.14		4481.14	

The salary scale is $s_y = 1.05^y$. *The accrual rate is 0.02. The benefit for employees who withdraw is a deferred annual pension with COLA 2%, starting from age 65. For an individual aged 65, we have* $\ddot{a}_{65} = 12.85$. *The lifetable for an individual who has withdrawn is*

x	l_x	d_x
57	10000.00	7.54
58	9992.46	8.22
59	9984.24	8.95
60	9975.29	9.76
61	9965.52	10.65
62	9954.87	11.63
63	9943.25	12.69
64	9930.55	13.86
65	9916.69	15.15

Calculate the EPV of accrued deferred pension benefits made to an individual aged exactly 57, with 16 years of service, whose salary for the past year was $121,000.

The current final average salary is $121,0001 + 1.05^{-1} + 1.05^{-2}3 = 115329.55$, so if the individual withdraws at age t, then the annual accrued pension benefits are $0.32\times 115329.55(1.05)^t(1.02)^{22-t} = 57055.09\left(\frac{1.05}{1.02}\right)^t$.

The total EPV of accrued deferred pension benefits is

$$12.85\times 57055.09\times \frac{1}{8588.36}\times 9916.69\left(49.18\times\frac{1}{9996.23}\times\left(\frac{1.05}{1.02}\right)^{0.5}+46.37\times\frac{1}{9988.35}\times\left(\frac{1.05}{1.02}\right)^{1.5}+43.62\times\frac{}{99}\right.$$