ACSC/STAT 4720, Life Contingencies II
Fall 2016
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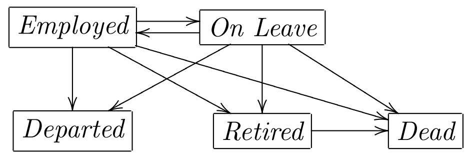
Homework Sheet 1
Model Solutions

Basic Questions

1. *An insurance company is developing a new policy. The policy considers 4 states: Employed, On leave, Retired, and Dead. The transition diagram is shown below:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) Employed—Departed—On leave—Dead

Impossible — transitions from "Departed" to "On leave" are not permitted.

(ii) Employed—Departed—Dead

Impossible — transitions from "Departed" to "Dead" are not permitted.

(iii) Employed—On leave—Employed—Retired

Possible.

(iv) Employed—On leave—Dead—Retired

Impossible — transitions from "Dead" to "Retired" are not possible.

(v) Employed—On leave—Retired—Departed

Impossible — transitions from "Retired" to "Departed" are not posssible.

2. *Consider a permanent disability model with transition intensities*

$$\mu_x^{01} = 0.004 + 0.000003x$$
$$\mu_x^{02} = 0.001 - 0.000001x$$
$$\mu_x^{12} = 0.002 + 0.000004x$$

where State 0 is employed, State 1 is retired and State 2 is dead.

(a) Calculate the probability that an employed individual aged 31 is still employed at age 44.

The rate of exit from state 0 is $0.005 + 0.000002x$, so the probability of remaining in that state from age 31 to age 44 is

$$e^{-\int_{31}^{44}(0.005+0.000002x)\,dx} = e^{-[0.005x+0.000001x^2]_{31}^{44}} = e^{-(0.005\times 13+0.000001\times(44^2-31^2))} = e^{-0.065975} = 0.9361543$$

(b) Calculate the probability that an employed individual aged 31 is dead by age 42.

There are two ways for this to happen: the individual can die directly from employment, or the individual can retire first. The probability that the individual is still employed t years later is

$$e^{-(0.005t+0.000001(t^2+62t))} = e^{-(0.005062t+0.000001t^2)}$$

Now the probability that the individual dies directly from the employed state during the next 11 years is therefore

$$\begin{aligned}
&\int_0^{11}(0.004+0.000003(31+t))e^{-(0.005062t+0.000001t^2)}\,dt\\
&=\int_0^{11}(0.004093+0.000003t)e^{-(0.005062t+0.000001t^2)}\,dt\\
&=\int_0^{11}\left(3(0.002531+0.000001t)-0.0035\right)e^{-(0.005062t+0.000001t^2)}\,dt\\
&=\int_0^{11}\left(3(0.002531+0.000001t)-0.0035\right)e^{-(0.005062t+0.000001t^2)}\,dt\\
&=3\int_0^{11}(0.002531+0.000001t)e^{-(0.005062t+0.000001t^2)}\,dt-0.0035\int_0^{11}e^{-0.000001(5062t+t^2)}\,dt\\
&=1.5\left[-e^{-(0.005062t+0.000001t^2)}\right]_0^{11}-3.5\sqrt{\pi}\int_0^{11}\frac{e^{-0.000001((2531+t)^2-2531^2)}}{1000\sqrt{\pi}}\,dt\\
&=1.5\left(1-e^{-(0.005062\times 11+0.000001\times 11^2)}\right)-3.5\sqrt{\pi}e^{6.405961}\int_0^{11}\frac{e^{-\frac{(2531+t)^2}{1000^2}}}{1000\sqrt{\pi}}\,dt\\
&=1.5\left(1-e^{-0.055803}\right)-3.5\sqrt{\pi}e^{6.405961}\left(\Phi\left(\frac{11-2531}{500\sqrt{2}}\right)-\Phi\left(\frac{-2531}{500\sqrt{2}}\right)\right)\\
&=0.08141186-0.03959043\\
&=0.04182143
\end{aligned}$$

Conditional on the individual becoming disabled after t years, the probability of surviving to age 42 is

$$\begin{aligned}
e^{-\int_{31+t}^{42}(0.002+0.000004x)\,dx} &= e^{-[0.002x+0.000002x^2]_{31+t}^{42}}\\
&= e^{-(0.002(11-t)+0.000002(42^2-(31+t)^2))}\\
&= e^{-(11-t)(0.002+0.000002(73+t))}\\
&= e^{-(11-t)(0.002146+0.000002t)}
\end{aligned}$$

Therefore the probability that the individual becomes disabled and then dies before age 42 is

$$\int_0^{11}(0.002+0.000003t)e^{-(0.005062t+0.000001t^2)}\left(1-e^{-(11-t)(0.002146+0.000002t)}\right)\,dt$$

$$= \int_0^{11}(0.002+0.000003t)e^{-(0.005062t+0.000001t^2)}\,dt$$

$$- \int_0^{11}(0.002+0.000003t)e^{-(0.005062t+0.000001t^2)-(11-t)(0.002146+0.000002t)}\,dt$$

$$= 1.5(1-e^{-0.055803}) - 5.593\sqrt{\pi}e^{6.405961}\left(\Phi\left(\frac{11-2531}{500\sqrt{2}}\right)-\Phi\left(\frac{-2531}{500\sqrt{2}}\right)\right)$$

$$- \int_0^{11}(0.002+0.000003t)e^{-0.000001(23606+2938t-t^2)}\,dt$$

$$= 0.01814635 - \int_0^{11}(0.002+0.000003t)e^{0.000001(t-1469)^2-2.181567}\,dt$$

$$= 0.01814635 - e^{-2.181567}\int_0^{11}0.000003\left(t-1496+\left(1496+\frac{2000}{3}\right)\right)e^{0.000001(t-1469)^2}\,dt$$

$$= 0.01814635 - e^{-2.181567}\left(0.000003\int_0^{11}(t-1496)\,e^{0.000001(t-1469)^2}\,dt + 0.006488\int_0^{11}e^{0.000001(t-1469)^2}\,dt\right)$$

$$= 0.01814635 - e^{-2.181567}\left(1.5\left[e^{0.000001(t-1469)^2}\right]_0^{11} + 0.006488\times 1000\sqrt{\pi}\left(\Phi\left(-1.458\sqrt{2}\right)-\Phi\left(-1.469\sqrt{2}\right)\right)\right)$$

$$= 0.01814635 - e^{-2.181567}\left(1.5\left(e^{(1.458)^2}-e^{(1.469)^2}\right)+0.008381697\right)$$

$$= 0.01814635 - e^{-2.181567}(0.008381697-0.4112676)$$

$$= 0.01814635 + 0.04547153$$

$$= 0.06361788$$

3. *Under a disability income model with transition intensities*

$$\mu_x^{01} = 0.001$$
$$\mu_x^{10} = 0.002$$
$$\mu_x^{02} = 0.003$$
$$\mu_x^{12} = 0.005$$

calculate the probability that a healthy individual dies within the next 4 years. [State 0 is healthy, State 1 is sick and State 2 is dead.]

We sum over the number of transitions to or from the sick state before death. The probability of direct death during 4 years is

$$\int_0^4 0.003e^{-0.004t}\,dt = \left[-\frac{0.003}{0.004}e^{-0.004t}\right]_0^4 = \frac{3(1-e^{-0.016})}{4} = 0.01190451$$

If the life becomes disabled at time t, then the probability of dying within the 4 years without recovering is

$$\int_t^4 0.005e^{-0.007s}\,ds = \left[-\frac{0.005}{0.007}e^{-0.007s}\right]_t^4 = \frac{5(e^{-0.007t} - e^{-0.028})}{7}$$

The total probability of becoming disabled then dying within the 4 years is therefore

$$\begin{aligned}
\frac{5}{7}\int_0^4 0.001e^{-0.004t}(e^{-0.007t} - e^{-0.028})\,dt &= \frac{5}{7}\int_0^4 0.001(e^{-0.011t} - e^{-0.032})\,dt \\
&= \frac{5}{7}\left(\int_0^4 0.001e^{-0.011t}\,dt - \int_0^4 0.001e^{-0.032}\,dt\right) \\
&= \frac{5}{7}\left(\frac{1-e^{-0.044}}{11} - \frac{1-e^{-0.128}}{32}\,dt\right) \\
&= 0.0001133534
\end{aligned}$$

The probability of becoming sick, recovering then dying all within 4 years is given by

$$\begin{aligned}
&\int_0^4\int_0^t\int_0^s 0.001^2 \times 0.002 \times 0.003e^{-0.007(s-r)}e^{-0.004(t+r-s)}\,dr\,ds\,dt \\
&= 6\times 10^{-12}\int_0^4\int_0^t\int_0^s e^{-0.004t-0.003s+0.003r}\,dr\,ds\,dt \\
&= 6\times 10^{-12}\int_0^4 e^{-0.004t}\int_0^t e^{-0.003s}\int_0^s e^{0.003r}\,dr\,ds\,dt \\
&= 6\times 10^{-12}\int_0^4 e^{-0.004t}\int_0^t e^{-0.003s}\frac{e^{0.003s}-1}{0.003}\,dr\,ds\,dt \\
&= 6\times 10^{-12}\int_0^4 e^{-0.004t}\int_0^t \frac{1-e^{-0.003s}}{0.003}\,ds\,dt \\
&= 6\times 10^{-12}\int_0^4 e^{-0.004t}\left(\frac{t}{0.003} - \frac{1-e^{-0.003t}}{0.003^2}\right)dt \\
&= 6\times 10^{-12}\int_0^4 \left(\frac{te^{-0.004t}}{0.003} - \frac{e^{-0.004t}}{0.003^2} + \frac{e^{-0.007t}}{0.003^2}\right)dt \\
&= 6\times 10^{-12}\left(\left[-\frac{te^{-0.004t}}{0.004\times 0.003}\right]_0^4 + \int_0^4 \frac{e^{-0.004t}}{0.004\times 0.003}\,dt - \frac{1-e^{-0.016}}{0.003^2\times 0.004} + \frac{1-e^{-0.028}}{0.003^2\times 0.007}\right) \\
&= 6\times 10^{-12}\left(-\frac{4e^{-0.016}}{0.004\times 0.003} + \frac{1-e^{-0.016}}{0.004^2\times 0.003} - \frac{1-e^{-0.016}}{0.003^2\times 0.004} + \frac{1-e^{-0.028}}{0.003^2\times 0.007}\right) \\
&= 0.002\left(-0.0001e^{-0.016} + \frac{1-e^{-0.016}}{4^2} - \frac{1-e^{-0.016}}{3\times 4} + \frac{1-e^{-0.028}}{3\times 7}\right) \\
&= 6.304779\times 10^{-11}
\end{aligned}$$

Other terms are negligible, so the total probability is

$$0.01190451 + 0.0001133534 + 6.304779\times 10^{-11} = 0.01201786$$

4. *Under a critical illness model with transition intensities*

$$\mu_x^{01} = 0.001$$
$$\mu_x^{02} = 0.001$$
$$\mu_x^{12} = 0.005$$

calculate the premium for a 5-year policy with premiums payable continuously while the life is in the healthy state, which pays a death benefit of \$130,000 upon entry into state 2, and a benefit of \$50,000 upon entry into state 1, sold to a life in the healthy state (state 0). The interest rate is $\delta = 0.04$ *[State 0 is healthy, State 1 is sick and State 2 is dead.]*

The rate of exit from state 0 is $0.001 + 0.001 = 0.002$, so

$$\begin{aligned}\overline{a}^{00}_{x:\overline{5}|} &= \int_0^5 e^{-0.002t}e^{-0.04t}\,dt \\ &= \int_0^5 e^{-0.042t}\,dt \\ &= \left[-\frac{e^{-0.042t}}{0.042}\right]_0^5 \\ &= \frac{1-e^{-0.21}}{0.042} \\ &= 4.509899\end{aligned}$$

Also

$$\begin{aligned}\overline{A}^{01}_{x:\overline{5}|} &= \int_0^5 0.001e^{-0.002t}e^{-0.04t}\,dt \\ &= \int_0^5 0.001e^{-0.042t}\,dt \\ &= \left[-\frac{e^{-0.042t}}{42}\right]_0^5 \\ &= \frac{1-e^{-0.21}}{42} \\ &= 0.004509899\end{aligned}$$

For the death benefits, there are two ways the death benefit can be paid out — the life can directly die from the healthy state, or they can transition to the critically ill state first and then die. This gives

$$
\begin{aligned}
\overline{A}^{02}_{x:\overline{5}|} &= \int_0^5 0.001e^{-0.002t}e^{-0.04t}\,dt + \int_0^5\int_0^{5-t} 0.001e^{-0.002t}0.005e^{-0.005s}e^{-0.04(t+s)}\,ds\,dt\\
&= 0.004509899 + \int_0^5\int_0^{5-t} 0.000005e^{-0.042t}e^{-0.045s}\,ds\,dt\\
&= 0.004509899 + 0.000005\int_0^5 e^{-0.042t}\left[-\frac{e^{-0.045s}}{0.045}\right]_0^{5-t}\,dt\\
&= 0.004509899 + 0.000005\int_0^5 e^{-0.042t}\frac{(1-e^{-0.045(5-t)})}{0.045}\,dt\\
&= 0.004509899 + \frac{0.000005}{0.045}\int_0^5 \left(e^{-0.042t}-e^{0.003t-0.225}\right)\,dt\\
&= 0.004509899 + \frac{0.000005}{0.045}\left(\int_0^5 e^{-0.042t} - e^{-0.225}\int_0^5 e^{0.003t}\,dt\right)\\
&= 0.004509899 + \frac{0.005}{45}\left(\frac{1-e^{-0.21}}{0.042} - e^{-0.225}\frac{e^{0.015}-1}{0.003}\right)\\
&= 0.004509899 + \frac{0.005}{45}\left(\frac{1}{0.042} - e^{-0.21}\left(\frac{1}{0.042}+\frac{1}{0.003}\right)+\frac{e^{-0.225}}{0.003}\right)\\
&= 0.004509899 + \frac{1}{9}\left(\frac{1}{42} - e^{-0.21}\left(\frac{1}{42}+\frac{1}{3}\right)+\frac{e^{-0.225}}{3}\right)\\
&= 0.004509899 + \frac{1}{9}\left(\frac{1}{42} - \frac{5e^{-0.21}}{14}+\frac{e^{-0.225}}{3}\right)\\
&= 0.004509899 + 0.00005413591\\
&= 0.004564035
\end{aligned}
$$

The total EPV of the benefit is therefore $130000\times 0.004564035+50000\times 0.004509899 = \$818.82$. The premium is therefore $\frac{818.82}{4.509899} = \$181.56$.

5. *A whole life insurance policy can end either through death or withdrawl. The transition intensities are*

$$
\begin{aligned}
\mu^{01}_x &= 0.002 + 0.000003x\\
\mu^{02}_x &= 0.001 + 0.000004x
\end{aligned}
$$

Calculate the probability that an individual aged 43 withdraws from the policy before age 64. [State 0 is healthy, State 1 is withdrawn and State 2 is dead.]

The rate of leaving State 0 is $0.003+0.000007x$. The probability of remaining in State 0 for t years is therefore given by

$$
e^{-\int_{43}^{43+t}(0.003+0.000007x)\,dx} = e^{-\left[0.003x+0.0000035x^2\right]_{43}^{43+t}} = e^{-\left(0.003t+0.0000035(t^2+86t)\right)} = e^{-0.0000035\left(t^2+\frac{4602}{7}t\right)}
$$

The probability is given by

$$\int_0^{21} e^{-0.0000035\left(t^2+\frac{4602}{7}t\right)}(0.002+0.000003(43+t))\,dt$$
$$=\int_0^{21} e^{-0.0000035\left(t^2+\frac{4602}{7}t\right)}(0.002129+0.000003t)\,dt$$
$$=\int_0^{21} 0.000003e^{-0.0000035\left(t+\frac{2301}{7}\right)^2-0.0000035\left(\frac{2301}{7}\right)^2}\left(t+\frac{2129}{3}\right)dt$$
$$=0.000003e^{-0.3781858}\int_0^{21} e^{-0.0000035\left(t+\frac{2301}{7}\right)^2}\left(t+\frac{2301}{7}-\left(\frac{2301}{7}-\frac{2129}{3}\right)\right)dt$$
$$=0.000003e^{-0.3781858}\left(\int_0^{21}\left(t+\frac{2301}{7}\right)e^{-0.0000035\left(t+\frac{2301}{7}\right)^2}dt-\left(\frac{2301}{7}-\frac{2129}{3}\right)\int_0^{21} e^{-0.0000035\left(t+\frac{2301}{7}\right)^2}dt\right)$$
$$=0.000003e^{-0.3781858}\left(\frac{1-e^{-0.0000035\left(21^2+\frac{4602}{7}\times 21\right)}}{0.0000035}\right.$$
$$\left.+\left(\frac{8000}{21}\right)\left(\sqrt{\frac{\pi}{0.0000035}}\left(\Phi\left(\sqrt{0.000007}\times\frac{2448}{7}\right)-\Phi\left(\sqrt{0.000007}\times\frac{2301}{7}\right)\right)\right)\right)$$
$$=0.000003e^{-0.3781858}\left(13897.62+\frac{8000000}{21}\left(\sqrt{\frac{\pi}{3.5}}\left(\Phi\left(\frac{2.448}{\sqrt{7}}\right)-\Phi\left(\frac{2.301}{\sqrt{7}}\right)\right)\right)\right)$$
$$=0.000003e^{-0.3781858}(13897.62+5347.795)$$
$$=0.03955529$$

Standard Questions

6. *An insurance company is developing a new model for transition intensities in a disability income model. Under these transition intensities it calculates*

$$\begin{array}{lll}
\overline{A}_{39}^{02}=0.18 & \overline{A}_{44}^{02}=0.20 & \overline{A}_{44}^{12}=0.31\\
\overline{a}_{39}^{00}=17.47 & \overline{a}_{44}^{00}=17.33 & \overline{a}_{44}^{10}=0.17\\
\overline{a}_{39}^{01}=0.84 & \overline{a}_{44}^{01}=0.71 & \overline{a}_{44}^{11}=13.42\\
{}_5p_{39}^{00}=0.919 & {}_5p_{39}^{01}=0.026 & \delta=0.04
\end{array}$$

Calculate the premium for a 5-year policy for a life aged 39, with continuous premiums payable while in the healthy state, which pays a continuous benefit while in the sick state, at a rate of $40,000 per year, and pays a death benefit of $520,000 immediately upon death.

The EPV of the death benefit is

$$\begin{aligned}
520000\overline{A}^{02}{}^{\,1}_{39:\overline{5}|} &= 520000\left(\overline{A}^{02}_{39} - e^{-0.04\times 5}\left({}_5p^{00}_{39}\overline{A}^{02}_{44} +_5 p^{01}_{39}\overline{A}^{12}_{44}\right)\right) \\
&= 520000\left(0.18 - e^{-0.2}\left(0.919\times 0.20 + 0.026\times 0.31\right)\right) \\
&= \$11,917.53
\end{aligned}$$

The EPV of the disability benefit is

$$\begin{aligned}
40000\overline{a}^{01}_{39:\overline{5}|} &= 40000\left(\overline{a}^{01}_{39} - e^{-0.2}\left({}_5p^{00}_{39}\overline{a}^{01}_{44} +_5 p^{01}_{39}\overline{a}^{11}_{44}\right)\right) \\
&= 40000\left(0.84 - e^{-0.2}\left(0.919\times 0.71 + 0.026\times 13.42\right)\right) \\
&= \$804.59
\end{aligned}$$

Meanwhile we have

$$\begin{aligned}
\overline{a}^{00}_{39:\overline{5}|} &= \overline{a}^{00}_{39} - e^{-0.2}\left({}_5p^{00}_{39}\overline{a}^{00}_{44} +_5 p^{01}_{39}\overline{a}^{10}_{44}\right) \\
&= 17.47 - e^{-0.2}\left(0.919\times 17.33 + 0.026\times 0.17\right) \\
&= 4.427054
\end{aligned}$$

The annual rate of premium is therefore

$$\frac{11917.53 + 804.59}{4.427054} = \$2,873.72$$