ACSC/STAT 4720, Life Contingencies II

Fall 2016

Toby Kenney

Homework Sheet 2

Model Solutions

Basic Questions

1. *The following is a standard multiple decrement table giving probabilities of death and surrender for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.60	1.62
54	9958.78	39.43	1.74
55	9917.61	39.26	1.86
56	9876.49	39.09	2.00
57	9835.40	38.91	2.15

A life who is in poor health has the following lifetable.

x	l_x	d_x
53	10000.00	9.10
54	9990.90	9.90
55	9980.99	10.77
56	9970.22	11.72
57	9958.50	12.75

Use this lifetable and the standard multiple decrement table to produce a multiple decrement table for this life, assuming that this life has standard surrender probabilities, using:

(a) UDD in the multiple decrement table.

If the multiple decrement table satisfies UDD, and in a given year, the original decrement probabilities are q^{01} and q^{02}, then the intensity must satisfy $\mu_t^{01} = \frac{q^{01}}{1-t(q^{01}+q^{02})}$ and the individual decrement probability is

$$1 - e^{-\int_0^1 \frac{q^{01}}{1-t(q^{01}+q^{02})}\,dt} = 1 - e^{-\left[-\frac{q^{01}}{q^{01}+q^{02}}\log(1-t(q^{01}+q^{02}))\right]_0^1} = 1 - \left(1-(q^{01}+q^{02})\right)^{\frac{q^{01}}{q^{01}+q^{02}}}$$

Conversely, if the individual decrement probabilities are q' and q'', then the multiple decrement probabilities are the solutions to

$$q' = 1 - \left(1-(q^{01}+q^{02})\right)^{\frac{q^{01}}{q^{01}+q^{02}}}$$
$$q'' = 1 - \left(1-(q^{01}+q^{02})\right)^{\frac{q^{02}}{q^{01}+q^{02}}}$$

This gives

$$\frac{q^{01}}{q^{02}} = \frac{\log(1-q')}{\log(1-q'')}$$
$$1-(q^{01}+q^{02}) = (1-q')(1-q'')$$

If the original multidecrement probabilities are q^{01} and q^{02}, and the probability q^{02} is updated by q'', then the new multidecrement probabilities are the solution to

$$\frac{q^{01}_{\text{new}}}{q^{02}_{\text{new}}} = \frac{\frac{q^{01}}{q^{01}+q^{02}}\log(1-(q^{01}+q^{02}))}{\log(1-q'')}$$
$$1-(q^{01}_{\text{new}}+q^{02}_{\text{new}}) = (1-q'')(1-(q^{01}+q^{02}))^{\frac{q^{01}}{q^{01}+q^{02}}}$$

This gives the updated table

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.59	9.08
54	9951.33	39.38	9.85
55	9902.10	39.18	10.66
56	9852.25	38.98	11.56
57	9801.72	38.76	12.52

(b) UDD in the independent decrements.

If the independent decrements are q' and q'', then the intensities are given by $\mu'_t = \frac{q'}{1-tq'}$ and $\mu''_t = \frac{q''}{1-tq''}$. The multiple decrement probabilities are then the solutions to

$$\begin{aligned} q^{01}+q^{02} &= 1-e^{-\int_0^1 \frac{q'}{1-tq'}+\frac{q''}{1-tq''}\,dt} \\ &= 1-e^{-\left[-\log(1-tq')\right]_0^1-\left[-\log(1-tq'')\right]_0^1} \\ &= 1-(1-q')(1-q'') \\ \frac{q^{01}}{q^{02}} &= \frac{q'}{q''} \end{aligned}$$

Which gives:

$$\frac{q^{02}}{q''}(q'+q'') = 1-(1-q')(1-q'')$$
$$q^{02} = q''\frac{q'+q''-q'q''}{q'+q''}$$
$$q^{01} = q'\frac{q'+q''-q'q''}{q'+q''}$$

Given q^{01} and q^{02}, we can therefore solve for q' and q'' to get

$$(1-q'')\left(1-\frac{q^{01}}{q^{02}}q''\right) = 1-(q^{01}+q^{02})$$
$$q'' - \frac{q^{01}+q^{02}}{q^{01}}q'' + \frac{q^{02}(q^{01}+q^{02})}{q^{01}} = 0$$
$$q'' = \frac{\frac{q^{01}+q^{02}}{q^{01}} - \sqrt{\left(\frac{q^{01}+q^{02}}{q^{01}}\right)^2 - 4\left(\frac{q^{02}(q^{01}+q^{02})}{q^{01}}\right)}}{2}$$
$$= \frac{q^{01}+q^{02} - \sqrt{(q^{01}+q^{02})^2 - 4q^{01}q^{02}(q^{01}+q^{02})}}{2q^{01}}$$
$$q' = \frac{q^{01}+q^{02} - \sqrt{(q^{01}+q^{02})^2 - 4q^{01}q^{02}(q^{01}+q^{02})}}{2q^{02}}$$

Substituting in this value into the formulae for the new values of q^{01} and q^{02}, we get

$$q^{02}_{\text{new}} = q''\frac{q'+q''-q'q''}{q'+q''}$$
$$q^{01}_{\text{new}} = q'\frac{q'+q''-q'q''}{q'+q''}$$

This gives the multiple decrement table

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.58	9.09
54	9951.33	39.38	9.86
55	9902.09	39.17	10.68
56	9852.24	38.97	11.57
57	9801.71	38.75	12.54

2. *The mortalities for a husband and wife (whose lives are assumed to be independent) aged 34 and 51 respectively, are given in the following tables:*

x	l_x	d_x
34	*10000.00*	*2.50*
35	*9997.50*	*2.74*
36	*9994.76*	*3.01*
37	*9991.75*	*3.30*
38	*9988.45*	*3.62*
39	*9984.83*	*3.98*
40	*9980.85*	*4.37*
41	*9976.48*	*4.80*
42	*9971.69*	*5.26*
43	*9966.42*	*5.78*

x	l_x	d_x
51	*10000.00*	*7.70*
52	*9992.30*	*8.37*
53	*9983.93*	*9.11*
54	*9974.82*	*9.91*
55	*9964.91*	*10.78*
56	*9954.14*	*11.72*
57	*9942.42*	*12.74*
58	*9929.67*	*13.86*
59	*9915.81*	*15.07*
60	*9900.75*	*16.38*

The interest rate is $i = 0.07$.

(a) They want to purchase a 10-year joint life insurance policy with a death benefit of $400,000. Annual premiums are payable while both are alive. Calculate the net premium for this policy using the equivalence principle.

We calculate

$$\begin{aligned}\ddot{a}_{34,51:\overline{10}|} =&1 + 0.999750 \times 0.999230 \times (1.07)^{-1} + 0.999476 \times 0.998393 \times (1.07)^{-2} + 0.999175 \times 0.997482 \times (1.07)^{-3} + \\ &0.998845 \times 0.996491 \times (1.07)^{-4} + 0.998483 \times 0.995414 \times (1.07)^{-5} + 0.998085 \times 0.994242 \times (1.07)^{-6} + \\ &0.997648 \times 0.992967 \times (1.07)^{-7} + 0.997169 \times 0.991581 \times (1.07)^{-8} + 0.996642 \times 0.990075 \times (1.07)^{-9} \\ =&7.47711\end{aligned}$$

This gives

$$A_{34,51:\overline{10}|} = 1 - \frac{0.07}{1.07}\ddot{a}_{34,51:\overline{10}|} = 1 - \frac{0.07 \times 7.47711}{1.07} = 0.5108433$$

The probability that they are both alive at the end of 10 years is $0.996064 \times 0.988437 = 0.9845465$, so we have

$$A^{1}_{34,51:\overline{10}|} = A_{34,51:\overline{10}|} - 0.(1.07)^{-10} = 0.01034978$$

The annual premium is therefore

$$400000\frac{0.01034978}{7.47711} = \$553.68$$

(b) They want to purchase a 10-year reversionary annuity, which will provide an annuity to the husband of $40,000 at the start of each year if the husband is alive and the wife is dead, provided the wife dies within the first 10 years. Calculate the net premium for this policy using the equivalence principle. For the husband, $\ddot{a}_{44} = 13.96$.

The EPV of the benefits after the end of the 10 year period are $40000 \times 0.996642 \times 0.009925 \times 13.96 \times (1.07)^{-10} = \$2,807.87$.

At the start of each year during the policy, a payment is made if the husband is alive and the wife is dead. The EPV of all these payments is therefore

$$40000\left(0.999750\times(1-0.999230)\times(1.07)^{-1}+0.999476\times(1-0.998393)\times(1.07)^{-2}+0.999175\times(1-0.997482)\times(1.07)^{-3}+0.998845\times(1-0.996491)\times(1.07)^{-4}+0.998483\times(1-0.995414)\times(1.07)^{-5}+0.998085\times(1-0.994242)\times(1.07)^{-6}+0.997648\times(1-0.992967)\times(1.07)^{-7}+0.997169\times(1-0.991581)\times(1.07)^{-8}+0.996642\times(1-0.990075)\times(1.07)^{-9}\right)$$
$$=\$1,143.21$$

From part (a), we have $\ddot{a}_{34,51:\overline{10}|} = 7.47711$, so the annual premium is

$$\frac{2807.87+1143.21}{7.47711} = \$528.42$$

3. *A husband is 80; the wife is 67. Their lifetables while both are alive, and the lifetable for the wife if the husband is dead, are given below:*

x	l_x	d_x
80	*10000.00*	*589.06*
81	*9410.94*	*609.24*
82	*8801.70*	*626.21*
83	*8175.50*	*639.23*
84	*7536.26*	*647.59*
85	*6888.68*	*650.54*
86	*6238.14*	*647.43*
87	*5590.71*	*637.67*
88	*4953.04*	*620.87*
89	*4332.17*	*596.80*

x	l_x	d_x
67	*10000.00*	*117.26*
68	*9882.74*	*126.08*
69	*9756.65*	*135.43*
70	*9621.23*	*145.30*
71	*9475.93*	*155.69*
72	*9320.24*	*166.61*
73	*9153.64*	*178.03*
74	*8975.61*	*189.92*
75	*8785.69*	*202.26*
76	*8583.43*	*214.99*

x	l_x	d_x
67	*10000.00*	*217.43*
68	*9782.57*	*233.54*
69	*9549.03*	*250.30*
70	*9298.73*	*267.63*
71	*9031.10*	*285.39*
72	*8745.70*	*303.46*
73	*8442.24*	*321.63*
74	*8120.61*	*339.70*
75	*7780.91*	*357.38*
76	*7423.53*	*374.38*

Calculate the annual premium for a 10-year term insurance policy sold to the wife with death benefit \$500,000. The interest rate is $i = 0.03$*. Use the UDD assumption for handling changes to the wife's mortality in the event of the husband's death.*

We want to construct an overall lifetable for the wife. If at age y, the wife has probability q_a of dying if the husband is alive throughout the year, probability q_d of dying if the husband is dead at the start of the year, and the husband has probability q_h of dying during the year, then the probability of the wife dying in a year when the husband is alive at the start of the year is calculated by conditioning on the husband's time of death. If the husband survives the year with probability $1-q_h$, then the wife's probability of dying is q_a. If the husband dies at time t during the year, then the probability that the wife survives is

$$(1-tq_a)\left(\frac{1-q_d}{1-tq_d}\right) = (1-q_d)\left(1+t\frac{q_a-q_d}{1-tq_d}\right)$$

The husband's death is uniformly distributed over the year. We then integrate the above expression over t to get the probability that the wife survives the year conditional on the husband dying during the year:

$$\begin{aligned}\int_0^1 (1-q_d)\left(\frac{1-tq_a}{1-tq_d}\right)\,dt &= \frac{1-q_d}{q_d}\int_{1-q_d}^1 \left(\frac{1-(1-u)\frac{q_a}{q_d}}{u}\right)\,du \\ &= \frac{1-q_d}{{q_d}^2}\int_{1-q_d}^1 \left((q_d-q_a)u^{-1}+q_a\right)\,du \\ &= \frac{1-q_d}{{q_d}^2}\left[(q_d-q_a)\log(u)+q_a u\right]_{1-q_d}^1 \\ &= \frac{1-q_d}{{q_d}^2}\left((q_a-q_d)\log(1-q_d)+q_a q_d\right)\end{aligned}$$

[Where we made the substitution $u = 1 - tq_d$.]

The total probability of the wife surviving the year if the husband is alive at the start of the year is therefore

$$(1-q_h)(1-q_a)+q_h\frac{1-q_d}{{q_d}^2}\left((q_a-q_d)\log(1-q_d)+q_a q_d\right)$$

We can then calculate the wife's lifetable as follows:

	Husband alive at start			Husband dead at start		
age	P(both alive)	P(wife survives)	P(H dies W survives)	P(wife alive)	P(wife survives)	P(Wife alive)
67	1.0000000	0.9879768	0.05791808	0.00000000	0.9782570	1.0000000
68	0.9300587	0.9868797	0.05910415	0.05791808	0.9761269	0.9879768
69	0.8587519	0.9856767	0.05986811	0.11563954	0.9737879	0.9743915
70	0.7865837	0.9843580	0.06014918	0.17247650	0.9712187	0.9590602
71	0.7141307	0.9829112	0.05988556	0.22766157	0.9683992	0.9417923
72	0.6420415	0.9813202	0.05903205	0.28035284	0.9653018	0.9223944
73	0.5710163	0.9795706	0.05755099	0.32965716	0.9619023	0.9006734
74	0.5017998	0.9776447	0.05542364	0.37464898	0.9581682	0.8764487
75	0.4351582	0.9755201	0.05265728	0.41440038	0.9540697	0.8495586
76	0.3718483	0.9731742		0.44802413	0.9495685	0.8198725

The wife's lifetable is therefore:

x	l_x	d_x
67	10000.00	120.23
68	9879.77	135.85
69	9743.91	153.31
70	9590.60	172.68
71	9417.92	193.98
72	9223.94	217.21
73	9006.73	242.25
74	8764.49	268.90
75	8495.59	296.86
76	8198.72	325.70

From this, we get

$$\begin{aligned}\ddot{a}_{67:\overline{10}|} &= 1 + 0.9879768(1.03)^{-1} + 0.9743915(1.03)^{-2} + 0.9590602(1.03)^{-3} + 0.9417923(1.03)^{-4} + 0.9223944(1.03)^{-5} + \\ &\quad 0.9006734(1.03)^{-6} + 0.8764487(1.03)^{-7} + 0.8495586(1.03)^{-8} + 0.8198725(1.03)^{-9} \\ &= 7.019808\end{aligned}$$

and

$$\begin{aligned}A^{1}_{67:\overline{10}|} &= 0.01202319(1.03)^{-1} + 0.01358532(1.03)^{-2} + 0.01533133(1.03)^{-3} + 0.01726786(1.03)^{-4} + 0.01939794(1.03)^{-5} + \\ &\quad 0.02172093(1.03)^{-6} + 0.02422467(1.03)^{-7} + 0.02689012(1.03)^{-8} + 0.02968615(1.03)^{-9} + 0.03256963(1.03)^{-10} \\ &= 0.1766859\end{aligned}$$

The premium is therefore

$$\frac{500000 \times 0.1766859}{7.019808} = \$12,584.81$$

Standard Questions

4. *The following is a multiple decrement table giving probabilities of death and surrender for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.60	1.62
54	9958.78	39.43	1.74
55	9917.61	39.26	1.86
56	9876.49	39.09	2.00
57	9835.40	38.91	2.15
58	9794.34	38.74	2.31
59	9753.28	38.57	2.49
60	9712.22	38.39	2.69
61	9671.15	38.21	2.90
62	9630.03	38.02	3.14

A life insurance policy has a death benefit of $300,000 payable at the end of the year of death. Premiums are payable at the beginning of each year. Surrenders receive a payment equal to half the policy value (net policy values are calculated under the assumption that there are no surrenders (surrenders instead survive to the end of the year). Calculate the premium for a 10-year policy sold to a life aged 53 if the interest rate is $i = 0.06$.

If the surrenders survive to the end of the year, the new lifetable is

x	l_x	d_x
53	10000.00	1.62
54	9998.38	1.75
55	9996.63	1.87
56	9994.76	2.02
57	9992.73	2.18
58	9990.55	2.36
59	9988.19	2.55
60	9985.64	2.77
61	9982.88	2.99
62	9979.88	3.25

This gives $\ddot{a}_{53:\overline{10}|} = 7.795466$ and $A^{1}_{53:\overline{10}|} = 0.001657376$, so the net premium would be $\frac{300000\times 0.001657376}{7.795466} =$ \$63.78231.

The reserves are then given

$$_tV = 300000A^{1}_{(53+t):\overline{10-t}|} - 63.78231\ddot{a}_{(53+t):\overline{10-t}|}$$

and are given in the following table:

t	$_tV$	Surrender value
0	0.000	0.00
1	19.012	9.51
2	35.352	17.68
3	48.828	24.41
4	58.629	29.31
5	64.190	32.10
6	64.911	32.46
7	59.841	29.92
8	47.963	23.98
9	28.500	14.25

Now for the policy with surrenders we calculate expected premiums received and benefits as follows

Year	l_x	$d_x^{(1)}$	Surrender Value	$d_x^{(2)}$	EPV Total Payment	EPV Premium
53	10000.00	39.60	0.00	1.62	45.88457	1.0000000
54	9958.78	39.43	9.51	1.74	46.51984	0.9395075
55	9917.61	39.26	17.68	1.86	46.93123	0.8826638
56	9876.49	39.09	24.41	2.00	47.61639	0.8292491
57	9835.40	38.91	29.31	2.15	48.29147	0.7790558
58	9794.34	38.74	32.10	2.31	48.94240	0.7318901
59	9753.28	38.57	32.46	2.49	49.75652	0.6875685
60	9712.22	38.39	29.92	2.69	50.68994	0.6459188
61	9671.15	38.21	23.98	2.90	51.52739	0.6067799
62	9630.03	38.02	14.25	3.14	52.60079	0.5700006
Total					488.7605	7.672634

The premium is therefore given by

$$\frac{488.7605}{7.672634} = \$63.70$$

5. *A couple want to receive the following:*

- *While both are alive, they would like to receive a pension of \$60,000 per year.*
- *If the husband is alive and the wife is not, they would like to receive a pension of \$80,000 per year.*
- *If the wife is alive and the husband is not, they would like to receive a pension of \$50,000 per year.*
- *When the wife dies: if the husband is still alive, they would like a death benefit of \$130,000; otherwise, they would like a death benefit of \$85,000.*

Construct a combination of insurance and annuity policies that achieve this combination of benefits.

There are a number of possible solutions. Here is one example:

- A life annuity for the wife of \$50,000 per year.
- A joint life annuity of \$10,000 per year.
- A reversionary annuity of \$80,000 for the husband after the death of the wife.
- A contingent insurance policy of \$45,000 payable if the wife dies while the husband is still alive.
- A life insurance policy of \$85,000 on the wife.

This meets the needs of the couple as shown in the following table:

	Both alive	H alive, W dead	W alive, H dead	W dies, H alive	W dies H dead
life annuity (W)	\$50,000	\$0	\$50,000	\$0	\$0
joint life annuity	\$10,000	\$0	\$0	\$0	\$0
reversionary annuity	\$0	\$80,000	\$0	\$0	\$0
life insurance (W)	\$0	\$0	\$0	\$85,000	\$85,000
contingent insurance	\$0	\$0	\$0	\$0	\$45,000
Total	\$60,000	\$80,000	\$50,000	\$85,000	\$130,000

6. *A husband aged 53 and wife aged 56 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.0001y + 0.0004 \\
\mu_{xy}^{02} &= 0.0002x - 0.0006 \\
\mu_{xy}^{03} &= 0.0005 \\
\mu_{x}^{13} &= 0.0004x + 0.0008 \\
\mu_{y}^{23} &= 0.0002y - 0.0002
\end{aligned}$$

They want to purchase a last survivor insurance, which will pay a death benefit of \$700,000 as soon as they are both dead. Premiums are payable continuously while either is alive. Force of interest is $\delta = 0.05$.

(a) Calculate the annual rate of continuous premium.

We break down the benefits into three parts:

Both die together:

The expected benefit from them both dying together is

$$\begin{aligned}
&\int_0^\infty 0.0005e^{-\int_0^t (0.0005+0.0001(56+s)+0.0004+0.0002(53+s)-0.0006)\,ds}e^{-0.05t}\,dt\\
=&\int_0^\infty 0.0005e^{-\int_0^t (0.0165+0.0003s)\,ds}e^{-0.05t}\,dt\\
=&\int_0^\infty 0.0005e^{-\left[0.0165s+0.00015s^2\right]_0^t}e^{-0.05t}\,dt\\
=&\int_0^\infty 0.0005e^{-0.00015\left(t^2+110t\right)}e^{-0.05t}\,dt\\
=&\int_0^\infty 0.0005e^{-0.00015\left(t^2+\frac{1330}{3}t\right)}\,dt\\
=&\int_0^\infty 0.0005e^{-0.00015\left(t+\frac{665}{3}\right)^2+0.00015\left(\frac{665}{3}\right)^2}\,dt\\
=&0.0005e^{0.00015\left(\frac{665}{3}\right)^2}\int_0^\infty e^{-0.00015\left(t+\frac{665}{3}\right)^2}\,dt\\
=&0.0005e^{\frac{6.65^2}{6}}\sqrt{\frac{\pi}{0.00015}}\left(1-\Phi\left(\frac{665}{3}\sqrt{0.0003}\right)\right)\\
=&0.05e^{\frac{6.65^2}{6}}\sqrt{\frac{\pi}{1.5}}\left(1-\Phi\left(\frac{6.65}{\sqrt{3}}\right)\right)\\
&=0.007088018
\end{aligned}$$

Husband dies first:

Here we integrate first over the time at which the husband dies:

$$
\begin{aligned}
&\int_0^\infty e^{-0.05t}\int_0^t e^{-\int_0^s(\mu^{01}_{(53+r)(56+r)}+\mu^{02}_{(53+r)(56+r)}+\mu^{03}_{(53+r)(56+r)})\,dr}e^{-\int_s^t \mu^{23}_{(56+r)}\,dr}\mu^{02}_{(53+s)(56+s)}\mu^{23}_{(56+t)}\,ds\,dt\\
=&\int_0^\infty e^{-0.05t}\int_0^t e^{-\int_0^s(0.016+0.0003r)\,dr}e^{-\int_s^t 0.0002(56+r-1)\,dr}(0.0002(53+s)-0.0006)(0.0002(56+t-1))\,ds\,dt\\
=&4\times10^{-8}\int_0^\infty\int_0^t e^{-0.05t}e^{-(0.016s+0.00015s^2)}e^{-0.0001(110(t-s)+(t^2-s^2))}(50+s)(55+t)\,ds\,dt\\
=&4\times10^{-8}\int_0^\infty\int_0^t e^{-0.05t}e^{-0.0001(110t+t^2)}e^{-(0.0050s+0.00005s^2)}(50+s)(55+t)\,ds\,dt\\
=&4\times10^{-8}\int_0^\infty e^{-0.05t}e^{-0.0001(110t+t^2)}(55+t)\int_0^t e^{-0.00005((s+50)^2-50^2)}(50+s)\,ds\,dt\\
=&4\times10^{-4}e^{0.00005\times50^2}\int_0^\infty e^{-0.05t}e^{-0.0001(110t+t^2)}(55+t)\left[e^{-0.00005((s+50)^2)}\right]_0^t\,dt\\
=&4\times10^{-4}e^{0.00005\times50^2}\int_0^\infty e^{-0.05t}e^{-0.0001(110t+t^2)}(55+t)\left(e^{-0.00005\times50^2}-e^{-0.00005(50+t)^2}\right)\,dt\\
=&4\times10^{-4}\int_0^\infty e^{-0.0001(610t+t^2)}(55+t)\left(1-e^{-0.00005(100t+t^2)}\right)\,dt\\
=&4\times10^{-4}\int_0^\infty (55+t)\left(e^{-0.0001(610t+t^2)}-e^{-0.00005(1320t+3t^2)}\right)\,dt
\end{aligned}
$$

we have

$$
\begin{aligned}
\int_0^\infty (55+t)e^{-0.0001(610t+t^2)}\,dt &= 500-e^{9.3025}(305-55)\sqrt{10000\pi}\left(1-\Phi\left(\frac{0.061}{\sqrt{0.0002}}\right)\right)\\
&=5000-25000e^{9.3025}\sqrt{\pi}\left(1-\Phi\left(\frac{6.1}{\sqrt{2}}\right)\right)\\
&=1093.474
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^\infty (55+t)e^{-0.00005(1320t+3t^2)}\,dt &= \frac{10000}{3}-e^{7.26}(220-55)\sqrt{\frac{\pi}{0.00015}}\left(1-\Phi\left(\frac{0.066}{\sqrt{0.0003}}\right)\right)\\
&=\frac{10000}{3}-16500e^{7.26}\sqrt{\frac{\pi}{1.5}}\left(1-\Phi\left(\frac{6.6}{\sqrt{3}}\right)\right)\\
&=978.4296
\end{aligned}
$$

This gives that the EPV of benefits from cases where the husband dies first is

$$
\begin{aligned}
4\times10^{-4}\int_0^\infty (55+t)\left(e^{-0.0001(610t+t^2)}-e^{-0.00005(1320t+3t^2)}\right)\,dt &= 4\times10^{-4}\,(1093.474-978.4296)\\
&=0.04601776
\end{aligned}
$$

Wife dies first:

$$\int_0^{\infty} e^{-0.05t} \int_0^t e^{-\int_0^s (\mu^{01}_{(53+r)(56+r)} + \mu^{02}_{(53+r)(56+r)} + \mu^{03}_{(53+r)(56+r)})\,dr} e^{-\int_s^t \mu^{13}_{(53+r)}\,dr} \mu^{01}_{(53+s)(56+s)} \mu^{13}_{(56+t)}\,ds\,dt$$

$$= \int_0^{\infty} e^{-0.05t} \int_0^t e^{-\int_0^s (0.016+0.0003r)\,dr} e^{-\int_s^t 0.0004(53+r+2)\,dr} (0.0001(56+s)+0.0004)(0.0004(53+t+2))\,ds\,dt$$

$$= 4 \times 10^{-8} \int_0^{\infty} \int_0^t e^{-0.05t} e^{-(0.016s+0.00015s^2)} e^{-0.0002(110(t-s)+(t^2-s^2))} (60+s)(55+t)\,ds\,dt$$

$$= 4 \times 10^{-8} \int_0^{\infty} \int_0^t e^{-0.05t} e^{-0.0002(110t+t^2)} e^{(0.0060s+0.00005s^2)} (60+s)(55+t)\,ds\,dt$$

$$= 4 \times 10^{-8} \int_0^{\infty} e^{-0.05t} e^{-0.0002(110t+t^2)} (55+t) \int_0^t e^{0.00005((s+60)^2-60^2)} (60+s)\,ds\,dt$$

$$= 4 \times 10^{-4} e^{-0.00005\times 60^2} \int_0^{\infty} e^{-0.05t} e^{-0.0002(110t+t^2)} (55+t) \left[e^{0.00005((s+60)^2)} \right]_0^t dt$$

$$= 4 \times 10^{-4} e^{0.18} \int_0^{\infty} e^{-0.05t} e^{-0.0002(110t+t^2)} (55+t) \left(e^{0.00005(60+t)^2} - e^{0.00005\times 60^2} \right) dt$$

$$= 4 \times 10^{-4} \int_0^{\infty} e^{-0.0002(360t+t^2)} (55+t) \left(e^{0.00005(120t+t^2)} - 1 \right) dt$$

$$= 4 \times 10^{-4} \int_0^{\infty} (55+t) \left(e^{-0.00005(1320t+3t^2)} - e^{-0.0002(360t+t^2)} \right) dt$$

We already have that

$$\int_0^{\infty} (55+t) e^{-0.00005(1320t+3t^2)}\,dt = 978.4296$$

We can calculate

$$\begin{aligned}\int_0^{\infty} (55+t) e^{-0.0002(360t+t^2)}\,dt &= 2500 - e^{6.48}(180-55)\sqrt{\frac{\pi}{0.0002}} \left(1 - \Phi\left(\frac{0.072}{\sqrt{0.0004}}\right)\right) \\ &= 2500 - 12500 e^{6.48} \sqrt{\frac{\pi}{2}} \left(1 - \Phi\left(\frac{7.2}{2}\right)\right) \\ &= 874.8561\end{aligned}$$

This gives

$$\begin{aligned}&= 4 \times 10^{-4} \int_0^{\infty} (55+t) \left(e^{-0.00005(1320t+3t^2)} - e^{-0.0002(360t+t^2)} \right) dt = 0.0004(978.4296 - 874.8561) \\ &= 0.0414294\end{aligned}$$

The total expected benefit is $\overline{A}_{\overline{53,56}} = 0.007088018 + 0.04601776 + 0.0414294 = 0.09453518$. From this we calculate $\overline{a}_{\overline{53,56}} = \frac{1-0.09453518}{0.05} = 18.1093$. The annual rate of premium is therefore

$$\frac{700000 \times 0.09453518}{18.1093} = \$3,654.18$$

(b) Calculate the policy value after 5 years if the husband is dead and the wife is alive.

If the husband is dead and the wife is alive, then the policy is effectively a standard life insurance policy for a life aged 61 with mortality $\mu_y^{23} = 0.0002y - 0.0002$. This allows us to calculate

$$\begin{aligned}
\overline{A}_{61} &= \int_0^\infty e^{-0.05t}(0.0002(61+t) - 0.0002)e^{-\int_0^t (0.0002(61+s)-0.0002)\,ds}\,dt \\
&= \int_0^\infty 0.0002e^{-0.05t}(60+t)e^{-0.0001\left[120s+s^2\right]_0^t}\,dt \\
&= 0.0002\int_0^\infty (60+t)e^{-0.05t}e^{-0.0001\left(120t+t^2\right)}\,dt \\
&= 0.0002\int_0^\infty (60+t)e^{-0.0001\left(620t+t^2\right)}\,dt \\
&= 0.0002\left(5000 - e^{9.61}\,(310-60)\sqrt{\frac{\pi}{0.0001}}\left(1-\Phi\left(\frac{0.062}{\sqrt{0.0002}}\right)\right)\right) \\
&= 1 - 0.0002 \times 25000e^{9.61}\sqrt{\pi}\left(1-\Phi\left(\frac{6.2}{\sqrt{2}}\right)\right) \\
&= 1 - 5e^{9.61}\sqrt{\pi}\left(1-\Phi\left(\frac{6.2}{\sqrt{2}}\right)\right) \\
&= 0.2302303
\end{aligned}$$

From this, we get

$$\overline{a}_{61} = \frac{1-0.2302303}{0.05} = 15.39539$$

The policy value is therefore

$$0.2302303 \times 70000 - 15.39539 \times 3654.179 = \$104,903.70$$