ACSC/STAT 4720, Life Contingencies II
Fall 2017
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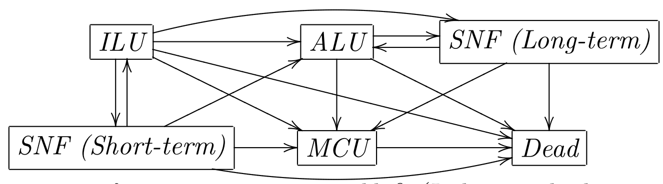
Homework Sheet 1
Model Solutions

Basic Questions

1. *An CCRC is developing a model for its care costs. The community has four levels of care: Independent Living Unit, Assisted Living Unit, Skilled Nursing Facility, and Memory Care Unit. The transition diagram is shown below:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) ILU—SNF (short-term)— ALU—Dead

Possible.

(ii) ILU—ALU—SNF (long-term)—ILU

Impossible — the transition from SNF (long-term) to ILU is not permitted.

(iii) ILU—ALU—MCU—Dead

Possible.

(iv) ILU—SNF (long-term)—MCU—ALU

Impossible — the transition from MCU to ALU is not permitted.

(v) ILU—MCU—ALU—Dead

Impossible — the transition from MCU to ALU is not permitted.

2. *Consider a permanent disability model with transition intensities*

$$\mu_x^{01} = 0.004 + 0.000001x$$
$$\mu_x^{02} = 0.001 + 0.000005x$$
$$\mu_x^{12} = 0.002 + 0.000003x$$

where State 0 is healthy, State 1 is permanently disabled and State 2 is dead.

(a) Calculate the probability that a healthy individual aged 22 is still healthy at age 41.

The individual must stay healthy for the intervening time. The probability of this is

$$\begin{aligned}
e^{-\int_{22}^{41}(\mu_x^{01}+mu_x^{02})\,dx} &= e^{-\int_{22}^{41}(0.005+0.000006x)\,dx}\\
&= e^{-\left[0.005x+0.000003x^2\right]_{22}^{41}}\\
&= e^{-\left(0.005(41-22)+0.000003(41^2-22^2)\right)}\\
&= 0.9061132
\end{aligned}$$

(b) Calculate the probability that a healthy individual aged 22 is dead by age 38.

There are two routes by which the individual can become dead — the probability that they die directly from the healthy state is

$$\begin{aligned}
\int_0^{16} {}_tp_{22+t}^{\overline{00}}\mu_{22+t}^{02}\,dt &= \int_0^{16} e^{-\left(0.005t+0.000003((22+t)^2-22^2)\right)}(0.001+0.000005(22+t))\,dt\\
&= \int_0^{16} e^{-\left(0.005132t+0.000003t^2\right)}(0.00111+0.000005t)\,dt\\
&= \int_0^{16} e^{-0.000003\left(t^2+\frac{5132}{3}t\right)}(0.001+0.000005(22+t))\,dt\\
&= \int_0^{16} e^{-0.000003\left(t+\frac{2566}{3}\right)^2+0.000003\left(\frac{2566}{3}\right)^2}(0.00111+0.000005t)\,dt\\
&= 0.000005e^{0.000003\left(\frac{2566}{3}\right)^2}\int_0^{16} e^{-0.000003\left(t+\frac{2566}{3}\right)^2}\left(t+\frac{2566}{3}-\frac{1900}{3}\right)\,dt\\
&= 0.000005e^{0.000003\left(\frac{2566}{3}\right)^2}\left[-\frac{e^{-0.000003\left(t+\frac{2566}{3}\right)^2}}{0.000006}-\frac{1900}{3}\sqrt{\frac{\pi}{0.000003}}\Phi\left(\sqrt{0.000006}\left(t+\frac{2566}{3}\right)\right)\right]_0^{16}\\
&= 0.01765192
\end{aligned}$$

The probability that the life first becomes disabled, then dies is

$$
\begin{aligned}
&\int_0^{16}\int_t^{16} {}_tp_{22+t}^{\overline{00}}\mu_{22+t}^{01}\,{}_{s-t}p_{22+t}^{\overline{11}}\mu_{22+s}^{12}\,ds\,dt\\
&=\int_0^{16}\int_t^{16} e^{-\left(0.005t+0.000003((22+t)^2-22^2)\right)}(0.004+0.000001(22+t))e^{-\left(0.002(s-t)+0.000003((22+s)^2-(22+t)^2)\right)}(0.002+0.000003\\
&=\int_0^{16}\int_t^{16} e^{-\left(0.003t+0.002132s+0.000003s^2\right)}(0.004022+0.000001t)(0.002+0.000003(22+s))\,ds\,dt\\
&=\int_0^{16}(0.002+0.000003(22+s))e^{-\left(0.002132s+0.000003s^2\right)}\int_0^s e^{-0.003t}(0.004022+0.000001t)\,dt\,ds\\
&=\int_0^{16}(0.002066+0.000003s)e^{-\left(0.002132s+0.000003s^2\right)}\left(\frac{0.004022}{0.003}(1-e^{-0.003s})+\frac{0.000001}{0.003}\left(\frac{1-e^{-0.003s}}{0.003}-se^{-0.003s}\right)\right)ds\\
&=\int_0^{16}(0.002066+0.000003s)e^{-\left(0.002132s+0.000003s^2\right)}\left(\frac{4.022}{3}(1-e^{-0.003s})+\frac{1-e^{-0.003s}}{9}-\frac{se^{-0.003s}}{3000}\right)ds\\
&=\int_0^{16}(0.002066+0.000003s)e^{-\left(0.002132s+0.000003s^2\right)}\left(\frac{13.066}{9}-e^{-0.003s}\left(\frac{13.066}{9}+\frac{s}{3000}\right)\right)ds\\
&=10^{-9}\int_0^{16}4355.333(688.6667+s)e^{-0.000003(s^2+710.667s)}-(2999373+5044s+s^2)e^{-0.000003(s^2+1710.667s)}\,ds\\
&=4.355\times10^{-6}e^{0.3787853}\left(\frac{e^{-0.3787853}-e^{-0.4136653}}{0.000006}-21.6667\sqrt{\frac{\pi}{0.0000003}}\left(\Phi(0.9095772)-\Phi(0.8703854)\right)\right)\\
&\quad-10^{-9}e^{2.194784}\left(\frac{2522e^{-2.194784}-2538e^{-2.277664}}{0.000003}-916665.4\sqrt{\frac{\pi}{0.000003}}\left(\Phi(2.134321)-\Phi(2.095129)\right)\right)\\
&=0.02448231
\end{aligned}
$$

The total probability of dying within 16 years is therefore $0.01765192+0.02448231=0.04213423$

3. *Under a disability income model with transition intensities*

$$
\begin{aligned}
\mu_x^{01}&=0.001\\
\mu_x^{10}&=0.002\\
\mu_x^{02}&=0.003\\
\mu_x^{12}&=0.005
\end{aligned}
$$

calculate the probability that a healthy individual dies within the next 4 years. [State 0 is healthy, State 1 is sick and State 2 is dead.]

The probability that a healthy individual dies without ever becoming disabled is

$$
\begin{aligned}
\int_0^4 {}_tp_x^{\overline{00}}\,\mu_{x+t}^{02}\,dt &= \int_0^4 0.003e^{-0.004t}\,dt \\
&= \frac{0.003}{0.004}(1-e^{-0.016}) \\
&= 0.01190451
\end{aligned}
$$

The probability that the individual becomes disabled then dies without recovering is

$$
\begin{aligned}
\int_0^4\int_t^4 {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{s-t}p_{x+t}^{\overline{11}}\mu_{x+s}^{12}\,ds\,dt &= \int_0^4\int_t^4 e^{-0.004t}e^{-0.007(s-t)}0.001\times 0.005\,ds\,dt \\
&= 5\times 10^{-6}\int_0^4\int_t^4 e^{0.003t}e^{-0.007s}\,ds\,dt \\
&= 5\times 10^{-6}\int_0^4 e^{-0.007s}\int_0^s e^{0.003t}\,dt\,ds \\
&= 5\times 10^{-6}\int_0^4 e^{-0.007s}\frac{e^{0.003s}-1}{0.003}\,ds \\
&= \frac{0.005}{3}\int_0^4 e^{-0.004s}-e^{-0.007s}\,ds \\
&= \frac{0.005}{3}\left(\frac{1-e^{-0.016}}{0.004}-\frac{1-e^{-0.028}}{0.007}\right) \\
&= 0.00000941826
\end{aligned}
$$

The probability that the individual becomes disabled then recovers once before dying is

$$\begin{aligned}
&\int_0^4\int_t^4\int_s^4 {}_tp_x^{\overline{00}}\mu_{x+t}^{01}{}_{s-t}p_{x+t}^{\overline{11}}\mu_{x+s}^{10}{}_{r-s}p_{x+s}^{\overline{00}}\mu_{x+r}^{02}\,dr\,ds\,dt\\
&=\int_0^4\int_t^4\int_s^4 e^{-0.004t}e^{-0.007(s-t)}e^{-0.004(r-s)}0.001\times 0.002\times 0.003\,ds\,dt\\
&=6\times 10^{-9}\int_0^4\int_t^4\int_s^4 e^{0.003t}e^{-0.003s}e^{-0.004r}\,dr\,ds\,dt\\
&=6\times 10^{-9}\int_0^4\int_0^r\int_0^s e^{0.003t}e^{-0.003s}e^{-0.004r}\,dt\,ds\,dr\\
&=6\times 10^{-9}\int_0^4 e^{-0.004r}\int_0^r e^{-0.003s}\int_0^s e^{0.003t}\,dt\,ds\,dr\\
&=2\times 10^{-6}\int_0^4 e^{-0.004r}\int_0^r e^{-0.003s}(e^{0.003s}-1)\,ds\,dr\\
&=2\times 10^{-6}\int_0^4 e^{-0.004r}\int_0^r (1-e^{-0.003s})\,ds\,dr\\
&=2\times 10^{-6}\int_0^4 e^{-0.004r}\left(r-\frac{1-e^{-0.003r}}{0.003}\right)dr\\
&=2\times 10^{-6}\left(\left[-\frac{re^{-0.004r}}{0.004}\right]_0^4+\int_0^4\frac{e^{-0.004r}}{0.004}\,dr-\int_0^4\frac{e^{-0.004r}}{0.003}\,dr+\int_0^4\frac{e^{-0.007r}}{0.003}\,dr\right)\\
&=2\left(\frac{1-e^{-0.028}}{21}-\frac{1-e^{-0.016}}{48}-0.001e^{-0.016}\right)\\
&=6.304779\times 10^{-08}
\end{aligned}$$

The other possibilities are much less likely, so the total probability is $0.01190451+0.00000941826+0.00000006304779 = 0.01191399$.

4. *Under a critical illness model with transition intensities*

$$\begin{aligned}
\mu_x^{01}&=0.001\\
\mu_x^{02}&=0.002\\
\mu_x^{12}&=0.12
\end{aligned}$$

calculate the premium for a 10-year policy with premiums payable continuously while the life is in the healthy state, which pays a death benefit of \$130,000 upon entry into state 2, and a benefit of \$80,000 upon entry into state 1, sold to a life in the healthy state (state 0). The interest rate is $\delta = 0.06$ [State 0 is healthy, State 1 is sick and State 2 is dead.]

We calculate

$$\begin{aligned}\overline{a}^{00}_{x:\overline{10|}} &= \int_0^{10} e^{-\delta t}\,{}_tp_x^{00}\,dt\\ &= \int_0^{10} e^{-0.06t}e^{-0.003t}\,dt\\ &= \int_0^{10} e^{-0.063t}\,dt\\ &= \frac{1-e^{-0.63}}{0.063}\\ &= 7.419178\end{aligned}$$

$$\begin{aligned}\overline{A}^{01}_{x:\overline{10|}} &= \int_0^{10} e^{-\delta t}\,{}_tp_x^{00}\mu^{01}_{x+t}\,dt\\ &= 0.001\overline{a}^{00}_{x:\overline{10|}}\end{aligned}$$

and

$$\begin{aligned}\overline{A}^{02}_{x:\overline{10|}} &= \int_0^{10} e^{-\delta t}\,{}_tp_x^{00}\mu^{02}_{x+t}\,dt + \int_0^{10}\int_t^{10} e^{-\delta s}\,{}_tp_x^{\overline{00}}\mu^{01}_{x+t}\,{}_{s-t}p_x^{\overline{11}}\mu^{12}_{x+s}\,ds\,dt\\ &= 0.002\overline{a}^{00}_{x:\overline{10|}} + 0.00012\int_0^{10} e^{-0.18s}\int_0^{s} e^{0.117t}\,dt\,ds\\ &= 0.002\overline{a}^{00}_{x:\overline{10|}} + 0.00012\int_0^{10} e^{-0.18s}\frac{e^{0.117s}-1}{0.117}\,ds\\ &= 0.002\overline{a}^{00}_{x:\overline{10|}} + \frac{0.00012}{0.117}\int_0^{10} e^{-0.063s}-e^{-0.18s}\,ds\\ &= 0.002\overline{a}^{00}_{x:\overline{10|}} + \frac{0.00012}{0.117}\left(\frac{1-e^{-0.63}}{0.063}-\frac{1-e^{-1.8}}{0.18}\right)\,ds\\ &= 0.002\overline{a}^{00}_{x:\overline{10|}} + 0.002853281\end{aligned}$$

The net premium is therefore

$$\frac{80000\times 0.001\overline{a}^{00}_{x:\overline{10|}} + 130000\left(0.002\overline{a}^{00}_{x:\overline{10|}} + 0.002853281\right)}{\overline{a}^{00}_{x:\overline{10|}}} = 80+260+\frac{370.9266}{7.419178} = \$390.00$$

5. *An employer offers a survivor benefit insurance policy. The possible exits from this policy are retirement, surrender, and death. The transition intensities are*

$$\mu_x^{01} = 0.002 + 0.000003x$$
$$\mu_x^{03} = 0.001 + 0.000004x$$
$$\mu_x^{02} = \begin{cases} 0 & \textit{if } x < 60 \\ 0.2(x-60) & \textit{if } x \geqslant 60 \end{cases}$$

Calculate the probability that an individual aged 34 withdraws from the policy before age 64. [State 0 is healthy, State 1 is surrender, State 2 is retired and State 3 is dead.]

This probability of surrenderring before age 60 is

$$\int_0^{26} {}_tp_{34}^{\overline{00}}\mu_{34+t}^{01}\,dt = \int_0^{26} e^{-(0.003t+\frac{0.000007}{2}((34+t)^2-34^2))}(0.002+0.000003(34+t))\,dt$$
$$= \int_0^{26} e^{-(0.003238t+0.0000035t^2)}(0.002102+0.000003t)\,dt$$
$$= 0.000003\int_0^{26} e^{-0.0000035(t^2+2\times 462.1429t)}(700.6667+t)\,dt$$
$$= 0.000003e^{0.7475162}\left(\left(\frac{e^{-0.7475162}-e^{-0.8339922}}{0.000007}\right) - 223.6191\sqrt{\frac{\pi}{0.0000035}}\left(\Phi\left(1.291505\right)-\Phi\left(1.222715\right)\right)\right)$$
$$= 0.0187878$$

This probability of surrenderring between age 60 and age 64 is

$$e^{-0.003\times 26+0.000007(60^2-34^2)}\int_0^{4} e^{-(0.003t+\frac{0.000007}{2}((60+t)^2-60^2)+0.1t^2}(0.002+0.000003(t+60))\,dt$$
$$= e^{-0.093288}\int_0^4 e^{-0.0000035\left(t^2+\left(\frac{0.006}{0.000007}+120\right)t\right)-0.1t^2}(0.00218+0.000003t)\,dt$$
$$= e^{-0.093288}\int_0^4 e^{-0.1000035\left(t^2+2\times 0.0170994t\right)}(0.00218+0.000003t)\,dt$$
$$= 0.000003e^{-0.093288}\int_0^4 e^{-0.1000035\left(t^2+2\times 0.0170994t\right)}(t+726.667)\,dt$$
$$= 0.000003e^{-0.09325876}\left(\frac{e^{-0.00002923997}-e^{-1.613765}}{0.200007} + 726.6325\sqrt{\frac{\pi}{0.1000035}}\left(\Phi\left(4.0170994\sqrt{0.200007}\right)-\Phi\left(0.0170994\sqrt{0\right.\right.\right.$$
$$= 0.005139104$$

The overall probability of surrendering is therefore $0.0187878 + 0.005139104 = 0.0239269$.

Standard Questions

6. *An insurance company is developing a new model for transition intensities in a disability income model. Under these transition intensities it calculates*

$$\overline{a}^{00}_{27} = 18.17 \qquad \overline{a}^{00}_{37} = 17.83 \qquad \overline{a}^{10}_{37} = 0.98$$
$$\overline{a}^{01}_{27} = 0.84 \qquad \overline{a}^{01}_{37} = 0.73 \qquad \overline{a}^{11}_{37} = 15.42$$
$$_{10}p^{00}_{27} = 0.919 \qquad _{10}p^{01}_{27} = 0.026 \qquad \delta = 0.05$$

Calculate the premium for a 10-year policy for a life aged 27, with continuous premiums payable while in the healthy state, which pays a continuous benefit while in the sick state, at a rate of \$80,000 per year, and pays a death benefit of \$900,000 immediately upon death. [Hint: to calculate A^{02}_x, consider how to extend the equation $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$ to the multiple state case by combining states 0 and 1.]

The premium is

$$\frac{80000\overline{a}^{01}_{27:\overline{10}|} + 900000\overline{A}^{02}_{27:\overline{10}|}}{\overline{a}^{00}_{27:\overline{10}|}}$$

We therefore just need to calculate the quantities involved in this expression. We have

$$\overline{a}^{00}_{27:\overline{10}|} = \overline{a}^{00}_{27} - e^{-0.5}\left({}_{10}p^{00}_{27}\overline{a}^{00}_{37} + {}_{10}p^{01}_{27}\overline{a}^{10}_{37}\right) = 18.17 - e^{-0.5}(0.919 \times 17.83 + 0.026 \times 0.98) = 8.216074$$

Similarly

$$\overline{a}^{01}_{27:\overline{10}|} = \overline{a}^{01}_{27} - e^{-0.5}\left({}_{10}p^{00}_{27}\overline{a}^{01}_{37} + {}_{10}p^{01}_{27}\overline{a}^{11}_{37}\right) = 0.84 - e^{-0.5}(0.919 \times 0.73 + 0.026 \times 15.42) = 0.1899265$$

We can calculate $\overline{A}^{02}_{27}$ because if we combine states 0 and 1, we have the equation $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$. Separating the states, this gives $\overline{a}^{00}_{27} + \overline{a}^{01}_{27} = \frac{1-\overline{A}^{02}_{27}}{\delta}$ so $\overline{A}^{02}_{27} = 1 - 0.05(18.17 + 0.84) = 0.0495$.
Similarly, $\overline{A}^{02}_{37} = 1 - 0.05(17.83 + 0.73) = 0.072$ and $\overline{A}^{12}_{37} = 1 - 0.05(0.98 + 15.42) = 0.18$. Now we have

$$\overline{A}^{02}_{27:\overline{10}|} = \overline{A}^{02}_{27} - e^{-10\delta}\left({}_{10}p^{00}_{27}\overline{A}^{02}_{37} + {}_{10}p^{01}_{27}\overline{A}^{12}_{37}\right) = 0.0495 - e^{-0.5}(0.919 \times 0.072 + 0.026 \times 0.18) = 0.006528516$$

The premium is therefore

$$\frac{80000 \times 0.1899265 + 900000 \times 0.006528516}{8.216074} = \$2564.46$$