ACSC/STAT 4720, Life Contingencies II

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Homework Sheet 2

Model Solutions

Basic Questions

1. *The following is a standard multiple decrement table giving probabilities of death and surrender for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.60	1.62
54	9958.78	39.43	1.74
55	9917.61	39.26	1.86
56	9876.49	39.09	2.00
57	9835.40	38.91	2.15

A life who is in poor health has the following lifetable.

x	l_x	d_x
53	10000.00	169.90
54	9830.10	189.00
55	9641.10	209.78
56	9431.32	232.25
57	9199.07	256.38

Use this lifetable and the standard multiple decrement table to produce a multiple decrement table for this life, assuming that this life has standard surrender probabilities, using:

(a) UDD in the multiple decrement table.

Since we have UDD, we have that ${}_tp_x^{00} = 1 - t + tp_x^{00}$. Recall that we have ${}_tp_x^{0i} = \int_0^t {}_sp_x^{\overline{00}}\mu_{x+s}^{0i}\,ds = tp_x^{0i}$, so we get ${}_tp_x^{\overline{00}}\mu_{x+t}^{0i} = p^{0i}$ for all t, so $\mu_{x+t}^{0i} = \frac{p^{0i}}{1-t+tp_x^{00}}$. Now in the single decrement model we have

$$p_x^{(i)} = e^{-\int_0^1 \mu_{x+t}^{0i}\,dt} = e^{-\int_0^1 \frac{p^{0i}}{1-t(1-p^{00})}\,dt} = e^{-\frac{p^{0i}}{1-p^{00}}\left[-\log\left(\frac{1}{1-p^{00}}-t\right)\right]_0^1} = e^{-\frac{p^{0i}}{1-p^{00}}\left(\log\left(\frac{1}{1-p^{00}}\right)-\log\left(\frac{1}{1-p^{00}}-1\right)\right)} = \left(p^{00}\right)^{\frac{p^{0i}}{1-p^{00}}}$$

This gives us the single decrement table for surrender:

$$p_{53}^{(1)} = (0.995878)^{\frac{39.60}{41.22}} = 0.9960397$$
$$p_{54}^{(1)} = \left(\frac{9917.61}{9958.78}\right)^{\frac{39.43}{41.17}} = 0.9960403$$
$$p_{55}^{(1)} = \left(\frac{9876.49}{9917.61}\right)^{\frac{39.26}{41.12}} = 0.996041$$
$$p_{56}^{(1)} = \left(\frac{9835.40}{9876.49}\right)^{\frac{39.09}{41.09}} = 0.9960417$$
$$p_{57}^{(1)} = \left(\frac{9794.34}{9835.40}\right)^{\frac{38.91}{41.06}} = 0.9960434$$

Now we combine the two single decrement tables using $p_x^{00} = \Pi p_x^{(i)}$ to get

$$p_{53}^{00} = p_{53}^{(1)} p_{53}^{(2)} = 0.9960397 \times \frac{9830.10}{10000.00} = 0.979117$$
$$p_{54}^{00} = p_{54}^{(1)} p_{54}^{(2)} = 0.9960403 \times \frac{9641.10}{9830.10} = 0.9768898$$
$$p_{55}^{00} = p_{55}^{(1)} p_{55}^{(2)} = 0.996041 \times \frac{9431.32}{9641.10} = 0.9743682$$
$$p_{56}^{00} = p_{56}^{(1)} p_{56}^{(2)} = 0.9960417 \times \frac{9199.07}{9431.32} = 0.9715138$$
$$p_{57}^{00} = p_{57}^{(1)} p_{57}^{(2)} = 0.9960434 \times \frac{8942.69}{9199.07} = 0.9682835$$

And also we have $p_x^{(i)} = \left(p_x^{00}\right)^{\frac{p_x^{0i}}{1-p_x^{00}}}$, so $p_x^{0i} = (1 - p_x^{00})\frac{\log(p_x^{(i)})}{\log(p_x^{00})}$, which gives us

$$p_{53}^{01} = (1 - p_{53}^{00})\frac{\log(p_{53}^{(i)})}{\log(p_{53}^{00})} = (1 - 0.979117)\frac{\log(0.9960397)}{\log(0.979117)} = 0.003926583$$
$$p_{53}^{01} = (1 - p_{53}^{00})\frac{\log(p_{53}^{(i)})}{\log(p_{53}^{00})} = (1 - 0.9768898)\frac{\log(0.9960403)}{\log(0.9768898)} = 0.003921536$$
$$p_{53}^{01} = (1 - p_{53}^{00})\frac{\log(p_{53}^{(i)})}{\log(p_{53}^{00})} = (1 - 0.9743682)\frac{\log(0.996041)}{\log(0.9743682)} = 0.003915799$$
$$p_{53}^{01} = (1 - p_{53}^{00})\frac{\log(p_{53}^{(i)})}{\log(p_{53}^{00})} = (1 - 0.9715138)\frac{\log(0.9960417)}{\log(0.9715138)} = 0.003909392$$
$$p_{53}^{01} = (1 - p_{53}^{00})\frac{\log(p_{53}^{(i)})}{\log(p_{53}^{00})} = (1 - 0.9682835)\frac{\log(0.9960434)}{\log(0.9682835)} = 0.003901241$$

This gives that the revised table is

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.27	169.56
54	9791.17	38.40	187.88
55	9564.89	37.45	207.71
56	9319.73	36.43	229.06
57	9054.24	35.32	251.84

(b) UDD in the independent decrements.

Under UDD in the independant decrements, we have

$$\mu_{x+t}^{0i} {}_tp_x^{(i)} = 1 - p_x^{(i)}$$

so $\mu_{x+t}^{0i} = \frac{1-p_x^{(i)}}{1-t(1-p_x^{(i)})}$

This gives

$$\begin{aligned}
{}_tp_x^{00} &= e^{-\int_0^t \sum \frac{1-p_x^{(i)}}{1-s(1-p_x^{(i)})}\,ds} \\
&= e^{-\sum\left(\int_0^t \frac{1}{\frac{1}{1-p_x^{(i)}}-s}\,ds\right)} \\
&= e^{-\sum\left(\log\left(\frac{1}{1-p_x^{(i)}}\right)-\log\left(\frac{1}{1-p_x^{(i)}}-t\right)\right)} \\
&= e^{\sum \log\left(1-t+tp_x^{(i)}\right)} \\
&= \prod (1-t+tp_x^{(i)})
\end{aligned}$$

and

$$\begin{aligned}
p_x^{0i} &= \int_0^1 tp_x^{00}\mu_{x+t}^{0i}\,dt \\
&= \int_0^1 \prod_j (1-t(1-p_x^{(j)}))\frac{1-p_x^{(i)}}{1-t(1-p_x^{(i)})}\,dt \\
&= (1-p_x^{(i)})\int_0^1 \prod_{j\neq i}(1-t(1-p_x^{(j)}))\,dt
\end{aligned}$$

For the two decrement case, this gives

$$
\begin{aligned}
p_x^{0i} &= (1-p_x^{(i)})\left[t - \frac{t^2}{2}(1-p_x^{(j)})\right]_0^1 \\
&= (1-p_x^{(i)})\left(1 - \frac{1-p_x^{(j)}}{2}\right) \\
&= \frac{(1-p_x^{(i)})(1+p_x^{(j)})}{2}
\end{aligned}
$$

Given the multiple decrement table, we will solve

$$
\begin{aligned}
p_x^{01} &= \frac{(1-p_x^{(1)})(1+p_x^{(2)})}{2} \\
p_x^{02} &= \frac{(1-p_x^{(2)})(1+p_x^{(1)})}{2} \\
p_x^{01} - p_x^{02} &= \frac{(1-p_x^{(1)})(1+p_x^{(2)})}{2} - \frac{(1-p_x^{(2)})(1+p_x^{(1)})}{2} \\
&= p_x^{(2)} - p_x^{(1)} \\
(p_x^{(2)} + p_x^{(1)})^2 &= (p_x^{(2)} - p_x^{(1)})^2 + 4p^{(1)}p^{(2)} \\
&= (p_x^{01} - p_x^{02})^2 + 4p^{00} \\
p_x^{(1)} &= \frac{p_x^{02} - p_x^{01} + \sqrt{(p_x^{01} - p_x^{02})^2 + 4p^{00}}}{2} \\
&= \frac{p_x^{02} - p_x^{01}}{2} + \sqrt{\left(\frac{p_x^{01} - p_x^{02}}{2}\right)^2 + p^{00}}
\end{aligned}
$$

From the multiple decrement table given we therefore calculate

$$p_{53}^{(1)} = \frac{1.62 - 39.60}{2 \times 10000.00} + \sqrt{\left(\frac{39.60 - 1.62}{2 \times 10000.00}\right)^2 + \frac{9958.78}{10000.00}} = 0.996039678578$$
$$p_{53}^{(1)} = \frac{1.62 - 39.43}{2 \times 9958.78} + \sqrt{\left(\frac{39.43 - 1.74}{2 \times 9958.78}\right)^2 + \frac{9917.61}{9958.78}} = 0.996034308245$$
$$p_{53}^{(1)} = \frac{1.62 - 39.26}{2 \times 9917.61} + \sqrt{\left(\frac{39.26 - 1.86}{2 \times 9917.61}\right)^2 + \frac{9876.49}{9917.61}} = 0.9960289133$$
$$p_{53}^{(1)} = \frac{1.62 - 39.09}{2 \times 9876.49} + \sqrt{\left(\frac{39.09 - 2.00}{2 \times 9876.49}\right)^2 + \frac{9835.40}{9876.49}} = 0.996022477$$
$$p_{53}^{(1)} = \frac{1.62 - 38.91}{2 \times 9835.40} + \sqrt{\left(\frac{38.91 - 2.15}{2 \times 9835.40}\right)^2 + \frac{9794.34}{9835.40}} = 0.996016505508$$

as in (a), we use $p_x^{00} = \Pi p_x^{(i)}$ to get

$$p_{53}^{00} = p_{53}^{(1)} p_{53}^{(2)} = 0.996039678578 \times \frac{9830.10}{10000.00} = 0.979116964439$$
$$p_{54}^{00} = p_{54}^{(1)} p_{54}^{(2)} = 0.996034308245 \times \frac{9641.10}{9830.10} = 0.976883894286$$
$$p_{55}^{00} = p_{55}^{(1)} p_{55}^{(2)} = 0.9960289133 \times \frac{9431.32}{9641.10} = 0.974356391966$$
$$p_{56}^{00} = p_{56}^{(1)} p_{56}^{(2)} = 0.996022477 \times \frac{9199.07}{9431.32} = 0.971495028002$$
$$p_{57}^{00} = p_{57}^{(1)} p_{57}^{(2)} = 0.996016505508 \times \frac{8942.69}{9199.07} = 0.968257317712$$

Finally, we calculate

$$p_{53}^{01} = \frac{1}{2}(1 - p_{53}^{(1)})(1 + p_{53}^{(2)}) = \frac{1}{2}(1 - 0.996039678578)\left(1 + \frac{9830.10}{10000.00}\right) = 0.00392667849152$$
$$p_{53}^{01} = \frac{1}{2}(1 - p_{53}^{(1)})(1 + p_{53}^{(2)}) = \frac{1}{2}(1 - 0.996034308245)\left(1 + \frac{9641.10}{9830.10}\right) = 0.00392756824956$$
$$p_{53}^{01} = \frac{1}{2}(1 - p_{53}^{(1)})(1 + p_{53}^{(2)}) = \frac{1}{2}(1 - 0.9960289133)\left(1 + \frac{9431.32}{9641.10}\right) = 0.00392788340537$$
$$p_{53}^{01} = \frac{1}{2}(1 - p_{53}^{(1)})(1 + p_{53}^{(2)}) = \frac{1}{2}(1 - 0.996022477)\left(1 + \frac{9199.07}{9431.32}\right) = 0.00392854895837$$
$$p_{53}^{01} = \frac{1}{2}(1 - p_{53}^{(1)})(1 + p_{53}^{(2)}) = \frac{1}{2}(1 - 0.996016505508)\left(1 + \frac{8942.69}{9199.07}\right) = 0.00392798408074$$

This gives that the revised table is

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
53	10000.00	39.27	179.56
54	9791.17	38.60	187.73
55	9564.84	37.86	207.42
56	9319.56	37.03	228.62
57	9053.91	36.11	251.29

2. *The mortalities for a husband and wife (whose lives are assumed to be independent) aged 42 and 33 respectively, are given in the following tables:*

x	l_x	d_x
42	10000.00	14.27
43	9985.73	15.66
44	9970.08	17.18
45	9952.90	18.86
46	9934.04	20.70
47	9913.34	22.72
48	9890.62	24.93
49	9865.69	27.37
50	9838.32	30.03
51	9808.29	32.95

x	l_x	d_x
33	10000.00	5.53
34	9994.47	6.04
35	9988.43	6.60
36	9981.83	7.22
37	9974.61	7.90
38	9966.71	8.65
39	9958.06	9.47
40	9948.59	10.37
41	9938.22	11.37
42	9926.85	12.46

The interest rate is $i = 0.04$.

(a) They want to purchase a 10-year joint life insurance policy with a death benefit of \$400,000. Annual premiums are payable while both are alive. Calculate the net premium for this policy using the equivalence principle.

We calculate

$$\ddot{a}_{42,33:\overline{10}|} = 1+0.998573\times0.999447(1.04)^{-1}+0.997008\times0.998843(1.04)^{-2}+0.995290\times0.998183(1.04)^{-3}+0.993404\times0.997461($$

so

$$A_{42,33:\overline{10}|} = 1 - \frac{0.04}{1.04} \times 8.34542253555 = 0.679022210171$$

and

$$A^{1}_{42,33:\overline{10}|} = 0.679022210171 - 0.977534 \times 0.991439(1.04)^{-10} = 0.024288838591$$

The premium is therefore $\frac{400000\times0.024288838591}{8.34542253555} = \$1164.18$

(b) They want to purchase a 10-year last survivor insurance with a benefit of \$8,000,000. Premiums are payable while either life is alive. Calculate the net premium for this policy using the equivalence principle.

The probability that at least one life is alive after t years is ${}_tp_{42} +_t p_{33} -_t p_{42t}p_{33}$, so we calculate

$$\ddot{a}_{\overline{42,33}:\overline{10}|} = 1+(0.998573+0.999447-0.998573\times0.999447)(1.04)^{-1}+(0.997008+0.998843-0.997008\times0.998843)(1.04)^{-2}+(0.9$$

so

$$A_{\overline{42,33:\overline{10}|}} = 1 - \frac{0.04}{1.04} \times 8.43502177583 = 0.675576085545$$

and

$$A^{1}_{\overline{42,33}:\overline{10}|} = 0.675576085545 - (0.977534 + 0.991439 - 0.977534 \times 0.991439)(1.04)^{-10} = 0.000141848939$$

The premium is therefore $\frac{8000000 \times 0.000141848939}{8.43502177583} = \$134.53$.

3. *A husband is 75; the wife is 88. Their lifetables while both are alive, and the lifetable for the husband if the wife is dead, are given below:*

x	l_x	d_x
75	10000.00	185.40
76	9814.60	205.34
77	9609.26	226.87
78	9382.39	249.99
79	9132.39	274.63
80	8857.77	300.63
81	8557.13	327.81
82	8229.33	355.83
83	7873.50	384.28
84	7489.22	412.59

x	l_x	d_x
88	10000.00	250.56
89	9749.44	273.99
90	9475.44	298.68
91	9176.76	324.47
92	8852.29	351.09
93	8501.20	378.22
94	8122.98	405.40
95	7717.58	432.08
96	7285.50	457.58
97	6827.93	481.09

x	l_x	d_x
75	10000.00	280.55
76	9719.45	307.75
77	9411.70	336.35
78	9075.35	366.07
79	8709.27	396.53
80	8312.74	427.22
81	7885.52	457.45
82	7428.07	486.43
83	6941.64	513.14
84	6428.50	536.44

Calculate the probability that the husband is alive in 10 years time. Use the UDD assumption for handling changes to the husband's mortality in the year of the wife's death.

Suppose in a particular year, the probability that the husband dies is q_a if the wife is alive, and q_d if the wife is dead. Then under the UDD assumption, the probability that the husband survives the year if the wife dies during the year is

$$\begin{aligned}\int_0^1 (1 - tq_a)\frac{1 - q_d}{1 - tq_d}\,dt &= (1 - q_d)\int_0^1 \frac{q_a}{q_d} - \frac{\frac{q_a}{q_d} - 1}{1 - tq_d}\,dt \\ &= (1 - q_d)\left(\frac{q_a}{q_d} + \left(\frac{q_a - q_d}{{q_d}^2}\right)\log(1 - q_d)\right)\end{aligned}$$

We therefore calculate:

Year	P(H survives year and wife alive at end of year)	P(H survives year and wife dead at start of year)
1	$\frac{9814.60}{10000.0} \times 0.974944 = 0.95686853824$	$\frac{9719.45}{10000.00} \times 0 = 0$
1	$\frac{9609.26}{9814.60} \times 0.947544 = 0.927719586885$	$\frac{9411.70}{9719.45} \times (1 - 0.974944) = 0.024262643997$
1	$\frac{9382.39}{9609.26} \times 0.917676 = 0.896010111667$	$\frac{9075.35}{9411.70} \times (1 - 0.947544) = 0.050581357204$
1	$\frac{9132.39}{9382.39} \times 0.885229 = 0.861641486584$	$\frac{8709.27}{9075.35} \times (1 - 0.917676) = 0.079003227807$
1	$\frac{8857.77}{9132.39} \times 0.850120 = 0.82455605076$	$\frac{8312.74}{8709.27} \times (1 - 0.885229) = 0.109545516736$
1	$\frac{8557.13}{8857.77} \times 0.812298 = 0.784727937702$	$\frac{7885.52}{8312.74} \times (1 - 0.850120) = 0.142177156702$
1	$\frac{8229.33}{8557.13} \times 0.771758 = 0.742194084014$	$\frac{7428.07}{7885.52} \times (1 - 0.812298) = 0.176813145505$
1	$\frac{7873.50}{8229.33} \times 0.728550 = 0.697048049477$	$\frac{6941.64}{7428.07} \times (1 - 0.771758) = 0.213295485487$
1	$\frac{7489.22}{7873.50} \times 0.682793 = 0.649468088075$	$\frac{6428.50}{6941.64} \times (1 - 0.728550) = 0.251383869662$
1	$\frac{7076.63}{7489.22} \times 0.634684 = 0.599718506723$	$\frac{5892.06}{6428.50} \times (1 - 0.682793) = 0.290736980076$

Year	P(H survives year and wife dies during year)
1	$\frac{9719.45}{10000.00}\left(\frac{1-\frac{9814.60}{10000.0}}{1-\frac{9719.45}{10000.00}}+\left(\frac{\frac{9719.45}{10000.00}-\frac{9814.60}{10000.0}}{\left(1-\frac{9719.45}{10000.00}\right)^2}\right)\log\left(\frac{9719.45}{10000.00}\right)\right)\times 0.025056 = 0.0244711271792$
1	$\frac{9411.70}{9719.45}\left(\frac{1-\frac{9609.26}{9814.60}}{1-\frac{9411.70}{9719.45}}+\left(\frac{\frac{9411.70}{9719.45}-\frac{9609.26}{9814.60}}{\left(1-\frac{9411.70}{9719.45}\right)^2}\right)\log\left(\frac{9411.70}{9719.45}\right)\right)\times 0.027399 = 0.0266770307948$
1	$\frac{9075.35}{9411.70}\left(\frac{1-\frac{9382.39}{9609.26}}{1-\frac{9075.35}{9411.70}}+\left(\frac{\frac{9075.35}{9411.70}-\frac{9382.39}{9609.26}}{\left(1-\frac{9075.35}{9411.70}\right)^2}\right)\log\left(\frac{9075.35}{9411.70}\right)\right)\times 0.029868 = 0.0289795156588$
1	$\frac{8709.27}{9075.35}\left(\frac{1-\frac{9132.39}{9382.39}}{1-\frac{8709.27}{9075.35}}+\left(\frac{\frac{8709.27}{9075.35}-\frac{9132.39}{9382.39}}{\left(1-\frac{8709.27}{9075.35}\right)^2}\right)\log\left(\frac{8709.27}{9075.35}\right)\right)\times 0.032447 = 0.0313572446171$
1	$\frac{8312.74}{8709.27}\left(\frac{1-\frac{8857.77}{9132.39}}{1-\frac{8312.74}{8709.27}}+\left(\frac{\frac{8312.74}{8709.27}-\frac{8857.77}{9132.39}}{\left(1-\frac{8312.74}{8709.27}\right)^2}\right)\log\left(\frac{8312.74}{8709.27}\right)\right)\times 0.035109 = 0.0337776537682$
1	$\frac{7885.52}{8312.74}\left(\frac{1-\frac{8557.13}{8857.77}}{1-\frac{7885.52}{8312.74}}+\left(\frac{\frac{7885.52}{8312.74}-\frac{8557.13}{8857.77}}{\left(1-\frac{7885.52}{8312.74}\right)^2}\right)\log\left(\frac{7885.52}{8312.74}\right)\right)\times 0.037822 = 0.0362024404645$
1	$\frac{7428.07}{7885.52}\left(\frac{1-\frac{8229.33}{8557.13}}{1-\frac{7428.07}{7885.52}}+\left(\frac{\frac{7428.07}{7885.52}-\frac{8229.33}{8557.13}}{\left(1-\frac{7428.07}{7885.52}\right)^2}\right)\log\left(\frac{7428.07}{7885.52}\right)\right)\times 0.040540 = 0.0385796659404$
1	$\frac{6941.64}{7428.07}\left(\frac{1-\frac{7873.50}{8229.33}}{1-\frac{6941.64}{7428.07}}+\left(\frac{\frac{6941.64}{7428.07}-\frac{7873.50}{8229.33}}{\left(1-\frac{6941.64}{7428.07}\right)^2}\right)\log\left(\frac{6941.64}{7428.07}\right)\right)\times 0.043208 = 0.0408482647106$
1	$\frac{6428.50}{6941.64}\left(\frac{1-\frac{7489.22}{7873.50}}{1-\frac{6428.50}{6941.64}}+\left(\frac{\frac{6428.50}{6941.64}-\frac{7489.22}{7873.50}}{\left(1-\frac{6428.50}{6941.64}\right)^2}\right)\log\left(\frac{6428.50}{6941.64}\right)\right)\times 0.045758 = 0.0429353819154$
1	$\frac{5892.06}{6428.50}\left(\frac{1-\frac{7076.63}{7489.22}}{1-\frac{5892.06}{6428.50}}+\left(\frac{\frac{5892.06}{6428.50}-\frac{7076.63}{7489.22}}{\left(1-\frac{5892.06}{6428.50}\right)^2}\right)\log\left(\frac{5892.06}{6428.50}\right)\right)\times 0.048109 = 0.0447567232751$

Year	P(H survives W a)	P(H survives W dead)	P(H survives W dies)	P(H survives year)
1	0.95686853824	0	0.0244711271792	0.981339665419
1	0.927719586885	0.024262643997	0.0266770307948	0.978659261677
1	0.896010111667	0.050581357204	0.0289795156588	0.97557098453
1	0.861641486584	0.079003227807	0.0313572446171	0.972001959008
1	0.82455605076	0.109545516736	0.0337776537682	0.967879221264
1	0.784727937702	0.142177156702	0.0362024404645	0.963107534869
1	0.742194084014	0.176813145505	0.0385796659404	0.957586895459
1	0.697048049477	0.213295485487	0.0408482647106	0.951191799675
1	0.649468088075	0.251383869662	0.0429353819154	0.943787339652
1	0.599718506723	0.290736980076	0.0447567232751	0.935212210074

The probability that the husband survives 10 years is therefore

$0.981339665419 \times 0.978659261677 \times 0.97557098453 \times 0.972001959008 \times 0.967879221264 \times 0.963107534869 \times 0.957586895459 \times 0.951191799675 \times 0.943787339652 \times 0.935212210074 = 0.682501221728$

Standard Questions

4. *The following is a multiple decrement table giving probabilities of accidental death and other death for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
44	*10000.00*	*1.03*	*1.37*
45	*9997.60*	*1.04*	*1.55*
46	*9995.01*	*1.06*	*1.74*
47	*9992.21*	*1.08*	*1.96*
48	*9989.17*	*1.09*	*2.23*
49	*9985.85*	*1.11*	*2.51*
50	*9982.23*	*1.13*	*2.83*
51	*9978.27*	*1.16*	*3.18*
52	*9973.93*	*1.18*	*3.59*
53	*9969.16*	*1.21*	*4.04*

A life insurance policy pays a benefit of \$300,000 at the end of the year of death, with an increased benefit of \$700,000 for accidental death. Premiums are payable at the beginning of each year. Calculate the premium for a 10-year policy sold to a life aged 44 if the interest rate is $i = 0.03$.

We calculate

$$\ddot{a}^{00}_{44:\overline{10}} = 1{+}0.999760(1.03)^{-1}{+}0.999501(1.03)^{-2}{+}0.999221(1.03)^{-3}{+}0.998917(1.03)^{-4}{+}0.998585(1.03)^{-5}{+}0.998223(1.03)^{-6}$$

$$\ddot{A}^{01}_{44:\overline{10}} = 0.000103(1.03)^{-1}{+}0.000104(1.03)^{-2}{+}0.000106(1.03)^{-3}{+}0.000108(1.03)^{-4}{+}0.000109(1.03)^{-5}{+}0.000111(1.03)^{-6}{+}$$

$$\ddot{A}^{02}_{44:\overline{10}} = 0.000137(1.03)^{-1}{+}0.000155(1.03)^{-2}{+}0.000174(1.03)^{-3}{+}0.000196(1.03)^{-4}{+}0.000223(1.03)^{-5}{+}0.000251(1.03)^{-6}{+}$$

The premium is therefore

$$\frac{700000 \times 0.000941900051688 + 300000 \times 0.00207195507487}{8.7748331621} = \$145.98$$

5. *A couple want to receive the following:*

   - *While both are alive, they would like to receive a pension of \$90,000 per year.*
   - *If the husband is alive and the wife is not, they would like to receive a pension of \$130,000 per year.*
   - *If the wife is alive and the husband is not, they would like to receive a pension of \$70,000 per year.*
   - *When the husband dies: if the wife is still alive, they would like a death benefit of \$30,000; otherwise, they would like a death benefit of \$80,000.*
   - *When the wife dies: if the husband is still alive, they would like a death benefit of \$60,000; otherwise, they would like a death benefit of \$120,000.*

   Construct a combination of insurance and annuity policies that achieve this combination of benefits.

   There are a number of possible solutions. Here is one example:

   - A life annuity for the husband of \$20,000 per year.

- A last survivor annuity of \$70,000 per year.
- A reversionary annuity of \$40,000 for the husband after the death of the wife.
- A last survivor life insurance of \$80,000.
- A contingent insurance policy of \$10,000 payable if the husband dies while the wife is still alive.
- A life insurance policy of \$60,000 on the wife.
- A life insurance policy of \$20,000 on the husband.

This meets the needs of the couple as shown in the following table:

	Both alive	H alive W dead	W alive H dead	W dies H alive	W dies H dead	H dies W alive	H dies W dead
life annuity (H)	\$20,000	\$20,000	\$0	\$0	\$0	\$0	\$0
last survivor annuity	\$70,000	\$70,000	\$70,000	\$0	\$0	\$0	\$0
reversionary annuity	\$0	\$40,000	\$0	\$0	\$0	\$0	\$0
life insurance (W)	\$0	\$0	\$0	\$60,000	\$60,000	\$0	\$0
life insurance (H)	\$0	\$0	\$0	\$0	\$0	\$20,000	\$20,000
contingent insurance	\$0	\$0	\$0	\$0	\$0	\$10,000	\$0
last survivor insurance	\$0	\$0	\$0	\$0	\$60,000	\$0	\$60,000
Total	\$90,000	\$130,000	\$70,000	\$60,000	\$120,000	\$30,000	\$80,000

6. *A husband aged 63 and wife aged 59 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.0002y + 0.0003 \\
\mu_{xy}^{02} &= 0.0002x - 0.0005 \\
\mu_{xy}^{03} &= 0.0008 \\
\mu_{x}^{13} &= 0.0003x + 0.0008 \\
\mu_{y}^{23} &= 0.0002y + 0.0004
\end{aligned}$$

They want to purchase a reversionary annuity, which will pay a continuous annuity of \$70,000 per year to the wife if the husband is dead. Premiums are payable continuously while both are alive. Force of interest is $\delta = 0.04$.

(a) Calculate the annual rate of continuous premium.

We calculate

$$
\begin{aligned}
{}_tp^{00} &= e^{-\int_0^t (0.025+0.0004t)\,dt} = e^{-0.025t-0.0002t^2} = \\
{}_tp^{02} &= \int_0^t {}_sp^{00}\mu^{02}_{63+s,59+s}\,{}_{t-s}p^{22}_{63+s,59+s}\,ds \\
&= \int_0^t e^{-0.0002(s^2+125s)}(0.0119+0.0002s)e^{-0.0251(t-s)-0.0002(t^2-s^2)}\,ds \\
&= \int_0^t e^{-0.0002(t^2+125.5t)+0.0001s}(0.0119+0.0002s)\,ds \\
&= e^{-0.0002(t^2+125.5t)}\left(0.0119\int_0^t e^{0.0001s}\,ds + 0.0002\int_0^t se^{0.0001s}\,ds\right) \\
&= e^{-0.0002(t^2+125.5t)}\left(119\left(e^{0.0001t}-1\right)+0.0002\left(\left[s\frac{e^{0.0001s}}{0.0001}\right]_0^t - \int_0^t \frac{e^{0.0001s}}{0.0001}\,ds\right)\right) \\
&= e^{-0.0002(t^2+125.5t)}\left(119\left(e^{0.0001t}-1\right)+2te^{0.0001t}-20000\left(e^{0.0001t}-1\right)\right) \\
&= 19881e^{-0.0002(t^2+125.5t)}+(2t-19881)e^{-0.0002(t^2+125t)}
\end{aligned}
$$

$$
\begin{aligned}
\overline{a}_{63,59} &= \int_0^\infty e^{-0.0002(t^2+125t)}e^{-0.04t}\,dt \\
&= \int_0^\infty e^{-0.0002(t^2+125t)}e^{-0.04t}\,dt \\
&= \int_0^\infty e^{-0.0002(t^2+325t)}\,dt \\
&= \sqrt{5000\pi}e^{\frac{162.5^2}{5000}}\int_0^\infty \frac{e^{-\frac{(t+162.5)^2}{5000}}}{\sqrt{5000}\pi}\,dt \\
&= \sqrt{5000\pi}e^{\frac{162.5^2}{5000}}\left(1-\Phi\left(\frac{162.5}{50}\right)\right) \\
&= \sqrt{5000\pi}e^{\frac{162.5^2}{5000}}\left(1-\Phi\left(\frac{162.5}{50}\right)\right) \\
&= 14.21911
\end{aligned}
$$

and

$$\begin{aligned}
\overline{a}_{63|59} &= \int_0^\infty (19881e^{-0.0002(t^2+125.5t)} + (2t-19881)e^{-0.0002(t^2+125t)})e^{-0.04t}\,dt \\
&= \int_0^\infty (19881e^{-0.0002(t^2+325.5t)} + (2t-19881)e^{-0.0002(t^2+325t)})\,dt \\
&= 19881\left(\int_0^\infty e^{-0.0002(t^2+325.5t)}\,dt - \int_0^\infty e^{-0.0002(t^2+325t)}\,dt\right) + 2\int_0^\infty te^{-0.0002(t^2+325t)}\,dt \\
&= 19881\left(e^{\frac{162.75^2}{5000}}\int_0^\infty e^{-\frac{(t+162.75)^2}{5000}}\,dt - e^{\frac{162.5^2}{5000}}\int_0^\infty e^{-\frac{(t^2+162.5)^2}{5000}}\,dt\right) + 2e^{\frac{162.5^2}{5000}}\int_0^\infty te^{-\frac{(t+162.5)^2}{5000}}\,dt \\
&= 19881e^{\frac{162.5^2}{5000}}\sqrt{5000\pi}\left(e^{\frac{162.75^2}{5000}-\frac{162.5^2}{5000}}\left(1-\Phi\left(\frac{162.75}{50}\right)\right) - \left(1-\Phi\left(\frac{162.5}{50}\right)\right)\right) + 2e^{\frac{162.5^2}{5000}}\int_0^\infty te^{-\frac{(t+162.5)^2}{5000}}\,dt \\
&= 2e^{\frac{162.5^2}{5000}}\int_0^\infty (t+162.5-162.5)e^{-\frac{(t+162.5)^2}{5000}}\,dt - 376.1888 \\
&= 2e^{\frac{162.5^2}{5000}}\int_0^\infty (t+162.5)e^{-\frac{(t+162.5)^2}{5000}}\,dt - 325e^{\frac{162.5^2}{5000}}\int_0^\infty e^{-\frac{(t+162.5)^2}{5000}}\,dt - 376.1888 \\
&= 2e^{\frac{162.5^2}{5000}}\left[-2500e^{-\frac{(t+162.5)^2}{5000}}\right]_0^\infty - 325e^{\frac{162.5^2}{5000}}\int_0^\infty e^{-\frac{(t+162.5)^2}{5000}}\,dt - 376.1888 \\
&= 5000 - 325e^{\frac{162.5^2}{5000}}\sqrt{5000\pi}\left(1-\Phi\left(\frac{162.5}{50}\right)\right) - 376.1888 \\
&= 2.601343
\end{aligned}$$

The premium is therefore $\frac{70000\times 2.601343}{14.21911} = \$12,806.29$

(b) Calculate the policy value after 5 years if the husband is dead and the wife is alive.

If the husband is dead and the wife is alive, then we have

$$\begin{aligned}
\overline{a}_{64} &= \int_0^\infty e^{-0.0132t-0.0001t^2}e^{-0.04t}\,dt \\
&= \int_0^\infty e^{-0.0001(t^2+532t)}\,dt \\
&= e^{\frac{266^2}{10000}}\int_0^\infty e^{-0.0001(t+266)^2}\,dt \\
&= e^{\frac{266^2}{10000}}\sqrt{10000\pi}\left(1-\Phi\left(\frac{266}{\sqrt{5000}}\right)\right) \\
&= 17.68182
\end{aligned}$$

So the policy value is $70000\times 17.68182 = \$1,237,727.40$.