ACSC/STAT 4720, Life Contingencies II
Fall 2017
Toby Kenney
Homework Sheet 4
Model Solutions

Basic Questions

1. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.15 \quad \sigma_k = 0.7 \quad K_{2017} = -5.40 \quad \alpha_{58} = -4.61 \quad \beta_{58} = 0.28$$

It estimates that its reserves are adequate in a given year provided $q(58,t) < 0.0022$. *Calculate the probability that its reserves are still adequate in 10 years' time. Use UDD to calculate the relation between* q_x *and* m_x.

We have that

$$K_{2027} = K_{2017} - 0.15 \times 10 + 0.7(Z_{2018} + \cdots + Z_{2027}) \sim N(-6.90, 4.9)$$

Under UDD we have

$$m(x,t) = \frac{q(x,t)}{1 - \frac{q(x,t)}{2}}$$

We want to calculate

$$P(q(58,2027) < 0.0022) = P\left(m(58,2027) < \frac{0.0022}{1 - \frac{0.0022}{2}}\right) = P\left(m(58,2027) < 0.00220242266493\right)$$

We have that $\log(m(58,2027)) = \alpha_{58} + \beta_{58}K_{2027}$, so we solve

$$\begin{aligned} m(58,2027) &< 0.00220242266493 \\ -4.61 + 0.28K_{2027} &< \log(0.00220242266493) = -6.11819731317 \\ K_{2027} &< \frac{4.61 - 6.11819731317}{0.28} = -5.38641897561 \end{aligned}$$

We have that K_{2027} is normally distributed with mean -6.90 and variance 4.9, so the probability that the reserves are adequate is $\Phi\left(\frac{6.90-5.38641897561}{\sqrt{4.9}}\right) =$ 0.7529386.

2. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$$K^{(1)}_{2017} = -3.05 \qquad K^{(2)}_{2017} = 0.29 \qquad c^{(1)} = -0.15 \qquad c^{(2)} = -0.02$$
$$\sigma_{k_1} = 0.6 \qquad \sigma_{k_2} = 0.08 \qquad \rho = 0.2 \qquad \overline{x} = 44$$

(a) Use this scale to calculate the median value of $q(33, 2029)$.

We have that

$$\begin{aligned}\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right) &= K^{(1)}_{2029} + K^{(2)}_{2029}(x-\overline{x}) \\ &= K^{(1)}_{2029} - 11K^{(2)}_{2029}\end{aligned}$$

We have that

$$K^{(1)}_{2029} = K^{(1)}_{2017} + 12c^{(1)} + \sigma_{k_1}\left(Z^{(1)}_{2017} + \cdots + Z^{(1)}_{2029}\right) = -3.05 - 12\times 0.15 + 0.6\left(Z^{(1)}_{2017} + \cdots + Z^{(1)}_{2029}\right)$$

and

$$K^{(2)}_{2029} = K^{(2)}_{2017} + 12c^{(2)} + \sigma_{k_1}\left(Z^{(2)}_{2017} + \cdots + Z^{(2)}_{2029}\right) = 0.29 - 12\times 0.02 + 0.08\left(Z^{(2)}_{2017} + \cdots + Z^{(2)}_{2029}\right)$$

This gives us

$$\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right) = -6.4 + 0.6\left(Z^{(1)}_{2017} + \cdots + Z^{(1)}_{2029}\right) - 0.88\left(Z^{(2)}_{2017} + \cdots + Z^{(2)}_{2029}\right)$$

Since $Z^{(1)}_t$ and $Z^{(2)}_t$ follow a multivariate normal distribution with mean 0, the median value of $\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right)$ is -6.4. This gives that the median value of $q(33, 2029)$ is $\frac{e^{-6.4}}{1+e^{-6.4}} = 0.001658801$.

(b) The insurance company sets its premium based on mortality that is 5% higher than this median. What is the probability that the mortality $q(33, 2029)$ *exceeds this loading? (That is, what is the probability that* $q(33, 2029)$ *is more than 5% higher than the median mortality?)*

The insurance company sets its premium based on mortality $q(33, 2029) = 1.05\times 0.001658801 = 0.00174174105$. This corresponds to $\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right) =$

-6.35112680293. We therefore want to calculate the probability that

$$\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right) > -6.35112680293$$

$$-6.4+0.6\left(Z^{(1)}_{2017}+\cdots+Z^{(1)}_{2029}\right)-0.88\left(Z^{(2)}_{2017}+\cdots+Z^{(2)}_{2029}\right) > -6.35112680293$$

$$0.6\left(Z^{(1)}_{2017}+\cdots+Z^{(1)}_{2029}\right)-0.88\left(Z^{(2)}_{2017}+\cdots+Z^{(2)}_{2029}\right) > 0.04887319707$$

Because $Z_t^{(1)}$ and $Z_t^{(2)}$ follow a multivariate normal distribution with mean 0, variances 1 and 1 and covariance 0.2, we have that $0.6Z_t^{(1)} - 0.88Z_t^{(2)}$ is normally distributed with mean 0 and variance $0.6^2+0.88^2-2\times 0.6\times 0.88\times 0.2 = 0.9232$. This gives that

$$P(q(33,2029) > 0.00174174105) = 1-\Phi\left(\frac{0.04887319707}{\sqrt{12\times 0.9232}}\right) = 0.4941423$$

Standard Questions

3. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c=-0.2 \qquad \sigma_k = 0.9 \qquad K_{2017}=-5.73$$

And the following values of α_x and β_x:

x	α_x	β_x
53	-4.578357	0.20040254
54	-4.354993	0.17848042
55	-4.963561	0.14949735
56	-5.472294	0.13804676
57	-5.645666	0.11540977
58	-6.967126	0.09578503

Using the approximation $m(x,t) \approx q(x,t)$, calculate the probability that a life aged 54 survives for two years under this model.

Given the values of $q(54,2018)$ and $q(55,2019)$, this probability is $(1-q(54,2018))(1-q(55,2019))$, so the overall probability is

$$\mathbb{E}\left((1-q(54,2018))(1-q(55,2019))\right)$$

Using the approximation $m(x,t)\approx q(x,t)$, we want to calculate

$$\mathbb{E}\left((1-m(54,2018))(1-m(55,2019))\right) = 1-\mathbb{E}(m(54,2018))-\mathbb{E}(m(55,2019))-\mathbb{E}(m(54,2018)m(55,2019))$$

We have that

$$\log(m(54, 2018)) = -4.354993 + 0.17848042K_{2018}$$

and

$$\log(m(55, 2019) = -4.963561 + 0.14949735K_{2019}$$

We also have that $K_{2018} = -5.73 - 0.2 + 0.9Z_{2018} = -5.93 + 0.9Z_{2018}$ and $K_{2019} = -6.13 + 0.9(Z_{2018} + Z_{2019})$, $m(54, 2018)$ follows a log-normal distribution with $\mu = -4.354993 + 0.17848042 \times (-5.93) = -5.4133818906$ and $\sigma = 0.9 \times 0.17848042 = 0.160632378$. The mean of this distribution is $e^{-5.4133818906+\frac{0.160632378^2}{2}} = 0.00451441120087$, and the mean of $m(55, 2019)$ is $e^{-4.963561-0.4\times0.14949735+0.9^2\times0.14949735^2} = 0.0067026193629$. Finally, we have

$$\begin{aligned}\log(m(54, 2018)m(55, 2019)) &= \log(m(54, 2018)) + \log(m(55, 2019))\\ &= -4.354993 + 0.17848042K_{2018} + -4.963561 + 0.14949735K_{2019}\\ &= -9.414049024 + 0.32797777Z_{2018} + 0.14949735Z_{2019}\end{aligned}$$

Thus $m(54, 2018)m(55, 2019)$ follows a log-normal distribution with $\mu = -9.414049024$ and $\sigma^2 = 0.32797777^2 + 0.14949735^2 = 0.129918875271$. This gives $\mathbb{E}(m(54, 2018)m(55, 2019)) = e^{-9.414049024+\frac{0.129918875271}{2}} = 0.0000870446296499$. The probability that the life survives two years is therefore

$$1-0.00451441120087-0.0067026193629+0.0000870446296499 = 0.988870014066$$

4. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$K^{(1)}_{2017} = -4.16$	$K^{(2)}_{2017} = 0.18$	$c^{(1)} = -0.1$	$c^{(2)} = 0.01$
$\sigma_{k_1} = 0.4$	$\sigma_{k_2} = 0.05$	$\rho = -0.4$	$\overline{x} = 48$

The company bases its long-term reserves on the median values of $q(42, 2031)$ and $q(63, 2031)$. The company expects to have life insurance policies on lives aged 42, and annuities for lives aged 63 in 2031. The payments at the end of 2031 are \$200,000 for each life aged 42 that dies during the year, and \$30,000 for each life aged 63 that survives the year. The company expects to have 2,000 life insurance policies for lives aged 42 and 8,000 annuities for lives aged 63. It sets the reserves for the year for each type of contract (that is, it sets reserves separately for life insurance and annuities) at 105% of the median payments under the CBD model. What is the probability that expected payments under the CBD model exceed this reserve for both lines of insurance? (That is, expected payments *for life insurance exceed the life insurance reserve, and expected payments for annuities exceed the annuity reserve.)*

Under the CBD model, we have that $K_{2031}^{(1)} \sim N(-4.61-14\times0.1, 14(0.4)^2) = N(-6.01, 2.24)$ while $K_{2031}^{(2)} \sim N(0.18+14\times0.01, 14(0.05)^2) \sim N(0.32, 0.07)$. Finally, we have $\mathrm{Cov}(K_{2031}^{(1)}, K_{2031}^{(2)}) = 14 \times 0.4 \times 0.05 \times -0.4 = -0.112$. The CBD model gives

$$\log\left(\frac{q(42, 2031)}{1-q(42, 2031)}\right) = K_{2031}^{(1)} - 6K_{2031}^{(2)}$$

Since $\log\left(\frac{p}{1-p}\right)$ is a monotone function of p, the median payments correspond to the median value $K_{2031}^{(1)} - 6K_{2031}^{(2)}$, which is equal to the mean, since this is normally distributed. That is, the median payments are based on $\log\left(\frac{q(42,2031)}{1-q(42,2031)}\right) = -6.01 - 6 \times 0.32 = -7.92$, so $q(42, 2031) = \frac{e^{-7.92}}{1+e^{-7.92}} = 0.000363270313217$.

The payments for the life insurance are proportional to $q(42, 2031)$, so the reserves correspond to $q(42, 2031) = 1.05 \times 0.000363270313217 = 0.000381433828878$. We therefore want to find the probability that $q(42, 2031) \geqslant 0.000381433828878$. This is equivalent to $\log\left(\frac{q(42,2031)}{1-q(42,2031)}\right) \geqslant \log\left(\frac{0.000381433828878}{1-0.000381433828878}\right) = -7.87119166555$

We have that $\log\left(\frac{q(42,2031)}{1-q(42,2031)}\right)$ follows a normal distribution with means -7.92 and variance $2.24+6^2\times0.07-2\times6\times(-0.112) = 6.104$. The probability that it exceeds -7.87119166555 is therefore $1-\Phi\left(\frac{-7.87119166555-(-7.95)}{\sqrt{6.104}}\right) = 1 - \Phi(0.031898105248) = 0.4872767$.