ACSC/STAT 4720, Life Contingencies II
Fall 2017
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Homework Sheet 5
Model Solutions

Basic Questions

1. *A disability income insurance company collects the following claim data (in thousands):*

i	d_i	x_i	u_i	i	d_i	x_i	u_i	i	d_i	x_i	u_i
1	0	1.9	-	8	0.5	0.6	-	15	2.0	-	5
2	0	-	5	9	0.5	1.3	-	16	2.0	4.4	-
3	0	2.1	-	10	0.5	0.7	-	17	2.0	4.5	-
4	0	0.3	-	11	0.5	2.5	-	18	2.0	3.9	-
5	0	0.1	-	12	1.0	3.5	-	19	5.0	6.3	-
6	0	0.1	-	13	1.0	-	5	20	5.0	7.0	-
7	0	2.1	-	14	1.0	5.0	-	21	5.0	7.9	-

Using a Kaplan-Meier product-limit estimator:

(a) estimate the probability that a random loss exceeds 3.4.

We compute

y_i	r_i	s_i
0.1	7	2
0.3	5	1
0.6	8	1
0.7	7	1
1.3	9	1
1.9	8	1
2.1	11	2
2.5	9	1

The Kaplan-Meier estimate is therefore

$$\frac{5}{7} \times \frac{4}{5} \times \frac{7}{8} \times \frac{6}{7} \times \frac{8}{9} \times \frac{9}{11} \times \frac{8}{9} = 0.277056277056$$

(b) estimate the median of the distribution.

We calculate the Kaplan-Meier estimates until the probability becomes less than 0.5. We get $\frac{5}{7} \times \frac{4}{5} \times \frac{7}{8} = 0.5$, so the median is 0.6.

(c) Use a Nelson-Åalen estimator to estimate the median of the distribution.

The median of the distribution corresponds to a cumulative hazard rate of $\log(2) = 0.6931472$. We calculate the partial sums of $\frac{s_i}{r_i}$ until it exceeds $\log(2)$. We get $\frac{2}{7} + \frac{1}{5} + \frac{1}{8} + \frac{1}{7} = 0.7535714$, so the median is 0.7.

2. *For the data in Question 1, use Greenwood's approximation to obtain a 95% confidence interval for the probability that a random loss exceeds 3.4, based on the Kaplan-Meier estimator.*

   (a) Using a normal approximation

   Greenwood's approximation gives

$$\begin{aligned}\mathrm{Var}(S_n(3.4)) &= (0.242424242424)^2 \left(\frac{2}{7\times 5} + \frac{1}{5\times 4} + \frac{1}{8\times 7} + \frac{1}{7\times 6} + \frac{1}{9\times 8} + \frac{1}{8\times 7} + \frac{2}{11\times 9} + \frac{1}{9\times 8}\right)\\ &= 0.0126146682621\end{aligned}$$

   The normal confidence interval is therefore $0.2424242424 \pm 1.96\sqrt{0.0126146682621} = [0.022286763571, 0.462561721277]$.

   (b) Using a log-transformed confidence interval.

   We have that $\mathrm{Var}(\log(-\log(S_n(3.4)))) = \frac{\mathrm{Var}(S_n(3.4))}{S_n(3.4)^2 \log(S_n(3.4))^2} = \frac{0.0126146682621}{0.2424242424^2 \log(0.2424242424)^2} = 0.106891598473$

   To find the log-transformed confidence interval, we therefore calculate $U = e^{-1.96\sqrt{0.106891598473}} = 0.526866596534$ and the interval is $\Big[0.2424242424242424^{\frac{1}{0.526866596534}}, 0.24242424242$ $[0.0679073725923, 0.473973080587]$.

3. *An insurance company records the following data in a mortality study:*

entry	*death*	*exit*	*entry*	*death*	*exit*	*entry*	*death*	*exit*
68.4	*71.0*	-	*69.8*	-	*73.7*	*69.1*	-	*72.1*
68.3	*71.4*	-	*68.4*	*72.8*	-	*68.6*	-	*72.3*
69.1	*73.8*	-	*68.7*	-	*71.4*	*71.0*	-	*71.9*
70.5	-	*72.6*	*70.0*	-	*72.1*	*70.3*	-	*71.0*
69.3	-	*72.8*	*70.3*	-	*72.0*	*68.6*	*72.1*	-
69.0	*73.1*	-	*70.6*	-	*73.1*	*68.7*	-	*72.6*
70.6	-	*71.3*	*70.2*	-	*71.3*	*69.7*	-	*73.8*
69.7	-	*72.4*	*71.0*	*72.9*	-	*70.6*	-	*73.5*
68.5	-	*72.3*	*69.2*	-	*71.8*	*70.7*	*72.3*	-
70.6	-	*71.4*	*70.4*	-	*71.7*	*69.6*	-	*72.3*
69.4	*71.4*	-	*68.3*	-	*73.4*	*68.2*	-	*72.8*

   Estimate the probability of an individual currently aged exactly 71 dying within the next year using:

   (a) the exact exposure method.

   The exact exposure is $0 + 0.4 + 1 + 1 + 1 + 1 + 0.3 + 1 + 1 + 0.4 + 0.4 + 1 + 1 + 0.4 + 1 + 1 + 1 + 0.3 + 1 + 0.8 + 0.7 + 1 + 1 + 1 + 0.9 + 0 +$

$1+1+1+1+1+1+1 = 26.6$. This gives the mortality rate as $\mu = \frac{3}{26.6} = 0.112781954887$, so the probability of dying during the year is $1 - e^{-0.112781954887} = 0.106654571514$.

(b) the actuarial exposure method.

The actuarial exposure is $1+1+1+1+1+1+0.3+1+1+1+0.4+1+1+0.4+1+1+1+0.3+1+0.8+0.7+1+1+1+0.9+0+1+1+1+1+1+1+1 = 28.8$. This gives the mortality rate as $q_{71} = \frac{3}{28.8} = 0.104166666667$.

4. *Using the following table:*

Age	*No. at start*	*enter*	*die*	*leave*	*No. at next age*
55	*0*	*28*	*4*	*10*	*14*
56	*14*	*31*	*3*	*14*	*28*
57	*28*	*21*	*6*	*24*	*19*
58	*19*	*38*	*1*	*42*	*14*
59	*14*	*29*	*2*	*41*	*0*

Estimate the probability that an individual aged 58 withdraws from the policy within the next year, conditional on surviving to the end of the year.

Assuming entries and deaths are uniformly distributed over the year, the actuarial exposure for policyholders aged 58 is $19 + \frac{38-1}{2} = \frac{75}{2}$. Of these, 42 leave, giving a probability of $\frac{42\times 2}{75} = 1.12$ which is obviously not reasonable. If instead, we use exact exposure, we get that the exposure is $19 + \frac{38-1-42}{2} = \frac{33}{2}$, and the estimated rate of withdrawl is $\frac{42\times 2}{33} = \frac{28}{11}$, so the probability of and individual withdrawing is $1 - e^{-\frac{28}{11}} = 0.9215626$.

5. *In a mortality study of 40 individuals in a disability income policy, an insurance company observes the following transitions.*

Entry	*State*	*Time*	*State*	*Time*	*State*	*Exit*
45.0	H					46.0
45.0	H					46.0
45.0	H					46.0
45.0	H					46.0
45.0	H					46.0
45.0	H					46.0
45.0	H					46.0
45.0	H					46.0
45.0	H	45.4	S			45.4
45.0	H	45.9	S			45.9
45.0	H	45.2	S			45.2
45.0	H	45.9	S			45.9
45.0	H	45.2	S			45.2
45.0	H	45.5	D			46.0
45.0	H	45.4	D			46.0
45.0	H	45.6	D			46.0
45.0	H	46.0	D			46.0
45.0	H	45.2	D	45.6	X	45.6
45.0	H	45.2	D	45.8	X	45.8
45.0	H	45.6	X			45.6
45.0	H	45.3	X			45.3
45.0	H	45.0	X			45.0
45.0	H	45.1	X			45.1
45.0	D					46.0
45.0	D					46.0
45.0	D					46.0
45.0	D	45.8	H			46.0
45.0	D	45.2	H			46.0
45.8	H					46.0
46.0	H					46.0
45.6	H					46.0
45.6	H					46.0
45.3	H					46.0
45.4	H					46.0
45.4	H	45.5	S			45.5
45.7	H	45.7	D	45.9	H	46.0
45.3	D					46.0
45.2	D					46.0
45.8	D					46.0
45.3	D	45.6	X			45.6
45.5	D	45.7	X			45.7

Based on these data, estimate the probability that an individual aged 45 who is disabled at the start of the year was healthy at some point during the year, but is disabled again at the end of the year.

We use exact exposure to estimate the transition intensities. The exposure in the healthy state is

$$1+1+1+1+1+1+1+1+0.4+0.9+0.2+0.9+0.2+0.5+0.4+0.6+0.0$$
$$+0.2+0.2+0.6+0.3+0.0+0.1+0+0+0+0.2+0.8+0.2+0.0+0.4$$
$$+0.4+0.7+0.6+0.1+0.1+0+0+0+0+0=17.0$$

The transition intensity from healthy to disabled is therefore $\frac{7}{17}=0.411764705882$. The transition intensity from healthy to surrendered or dead is $\frac{10}{17}=0.588235294118$.

The exposure in the disabled state is

$$0+0+0+0+0+0+0+0+0+0+0+0+0+0+0+0.5+0.6+0.4+0.0+0.4$$
$$+0.6+0+0+0+0+1+1+1+0.8+0.2+0+0+0+0+0+0+0+0.2$$
$$+0.7+0.8+0.2+0.3+0.2=8.9$$

The transition intensity from disabled to healthy is therefore $\frac{3}{8.9}=0.337078651685$. The transition intensity from disabled to dead is $\frac{4}{8.9}=0.4494382$.

The probability of transitioning from healthy to disabled and then back is therefore

$$\int_0^1\int_0^t 0.411764705882e^{-s}\times 0.337078651685e^{-0.786516853933(t-s)}e^{t-1}\,ds\,dt$$
$$=0.13879709187e^{-1}\int_0^1\int_0^t e^{0.213483146067(t-s)}\,ds\,dt$$
$$=0.13879709187e^{-1}\int_0^1 e^{0.213483146067t}\frac{1-e^{-0.213483146067t}}{0.213483146067}\,dt$$
$$=0.239178584043\int_0^1 e^{0.213483146067t}-1\,dt$$
$$=0.239178584043\left(\frac{e^{0.213483146067}-1}{0.213483146067}-1\right)$$
$$=0.0274483150305$$

Standard Questions

6. *For the study in Question 3, use the actuarial exposure method, and assume that the number of deaths follows a Poisson distribution with mean exposure times probability of dying to find a 95% confidence interval for q_{71}.*

Under the Poisson approximation, we have that $\mathrm{Var}(\widehat{q_{71}})=\frac{q_{71}}{e}$ so $\widehat{q_{71}}$ is approximately normally distributed with mean q_{71} and variance $\frac{q_{71}}{e}$. The

95% confidence interval then contains all solutions to

$$
\begin{aligned}
0.1016949 \in q_{71} &\pm 1.96\sqrt{\frac{q_{71}}{29.5}} \\
0 &\geqslant \left(0.1016949 - q_{71} - 1.96\sqrt{\frac{q_{71}}{29.5}}\right)\left(0.1016949 - q_{71} + 1.96\sqrt{\frac{q_{71}}{29.5}}\right) \\
0 &\geqslant (0.1016949 - q_{71})^2 - 1.96^2\,\frac{q_{71}}{29.5} \\
0 &\geqslant {q_{71}}^2 - 0.3336135q_{71} + 0.01034185 \\
q_{71} &\in \frac{0.3336135 \pm \sqrt{0.3336135^2 - 4 \times 0.01034185}}{2} \\
&= [0.034584808381, 0.299028691619]
\end{aligned}
$$