8.5 Numerical Evaluation of Probabilities

1

Density of event individual became disabled at time t is

$$ {}_tp^{00}_{27}\mu^{01}_{27+t}\ {}_{16-t}p^{11}_{27+t} $$

so probability is

$$ \int_0^{16} 0.003e^{-0.004t}e^{-0.002(16-t)}dt = 0.003e^{-0.032}\int_0^{16} e^{-0.002t}dt = \frac{0.003}{0.002}e^{-0.032}(1-e^{-0.032}) = 0.04575237 $$

2

Density of event individual became disabled at time t is

$$ {}_tp^{00}_{32}\mu^{01}_{32+t}\ {}_{12-t}p^{11}_{32+t} $$

We have

$$ \begin{aligned} {}_tp^{00}_x &= e^{-\int_0^t 0.004+0.00003(x+s)ds} = e^{-(0.004+0.00003x)t+0.000015t^2} \\ {}_tp^{11}_x &= e^{-\int_0^t 0.002+0.00002(x+s)ds} = e^{-(0.002+0.00002x)t+0.00001t^2} \end{aligned} $$

so probability is

$$ \begin{aligned} &\int_0^{12}(0.001+0.00001(32+t))e^{-(0.00496t+0.000015t^2)}e^{-(0.00264(12-t)+0.00001(12-t)^2)}dt \\ &= \int_0^{12}(0.001+0.00001(32+t))e^{-(0.00496t+0.000015t^2)}e^{-((0.00264+0.00002t)(12-t)+0.00001(12-t)^2)}dt \\ &= \int_0^{12}(0.001+0.00001(32+t))e^{-(0.03312+0.0028t+0.000005t^2)}dt \\ &= \int_0^{12}(0.001+0.00001(32+t))e^{-\frac{((t+28)^2)-121.6}{20000}}dt \\ &= \frac{e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}(132+t)e^{-\frac{((t+28)^2)}{20000}}dt \\ &= \frac{e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}(104+(t+28))e^{-\frac{((t+28)^2)}{20000}}dt \\ &= \frac{104e^{-\frac{121.6}{20000}}}{10000}\int_0^{12}e^{-\frac{((t+28)^2)}{20000}}dt + \frac{e^{-\frac{121.6}{20000}}}{10000}\left[-10000e^{-\frac{((t+28)^2)}{20000}}\right]_0^{12} \\ &= \frac{104}{100}e^{-\frac{121.6}{20000}}\sqrt{2\pi}\left(\Phi\left(\frac{28}{100}\right)-\Phi\left(\frac{16}{100}\right)\right)+e^{-\frac{121.6}{20000}}\left(e^{-\frac{28^2}{20000}}-e^{-\frac{40^2}{20000}}\right) \\ &= 0.1250398 \end{aligned} $$

3

We have

$$
\begin{aligned}
{}_tp_x^{\overline{00}} &= e^{-0.0004t} \\
{}_tp_x^{\overline{11}} &= e^{-0.00023t}
\end{aligned}
$$

Probability this happens from 1 transition:

$$
\begin{aligned}
&\int_0^{16} {}_tp_{27}^{\overline{00}}\mu_{27+t}^{01}\,{}_{(16-t)}p_{27+t}^{\overline{11}}\,dt \\
&= \int_0^{16} 0.0003e^{-0.0004t}e^{-0.00023(16-t)}\,dt \\
&= 0.0003e^{-0.00368}\int_0^{16} e^{-0.00017t}\,dt \\
&= \frac{0.0003}{0.00017}e^{-0.00368}\left(1-e^{-0.00272}\right)
\end{aligned}
$$

Probability this happens from 3 transitions:

$$
\begin{aligned}
&\int_0^{16}\int_s^{16}\int_t^{16} {}_sp_{27}^{\overline{00}}\mu_{27+s}^{01}\,{}_{(t-s)}p_{27+s}^{\overline{11}}\mu_{27+t}^{10}\,{}_{u-t}p_{27+t}^{\overline{00}}\mu_{27+u}^{01}\,{}_{(16-u)}p_{27+u}^{\overline{11}}\,du\,dt\,ds \\
&= \int_0^{16}\int_s^{16}\int_t^{16} 0.0003^2\times 0.00003e^{-0.0004(s+u-t)}e^{-0.00023(16-u+t-s)}\,du\,dt\,ds \\
&= 0.0003^2\times 0.00003e^{-0.00368}\int_0^{16}\int_s^{16}\int_t^{16} e^{-0.00017(s+u-t)}\,du\,dt\,ds
\end{aligned}
$$

Making the substitution $w = s+u-t$, this becomes (noting that $|J| = 1$):

$$
\begin{aligned}
&0.0003^2\times 0.00003e^{-0.00368}\int_0^{16}\int_0^{w}\int_s^{16+s-w} e^{-0.00017w}\,dt\,ds\,dw \\
&0.0003^2\times 0.00003e^{-0.00368}\int_0^{16} e^{-0.00017w}\int_0^{w}\int_s^{16+s-w} 1\,dt\,ds\,dw \\
&0.0003^2\times 0.00003e^{-0.00368}\int_0^{16} w(16-w)e^{-0.00017w}\,dw
\end{aligned}
$$

In general, the probability that this happens from $2n+1$ transitions is

$$
0.0003\left(0.00003\times 0.0003\right)^n e^{-0.00368}\int_0^{16}\frac{w^n(16-w)^n}{(n!)^2}e^{-0.00017w}\,dw
$$

The total probability of this is therefore

$$\sum_{n=0}^{\infty} 0.0003\,(0.00003\times 0.0003)^n\, e^{-0.00368}\int_0^{16}\frac{w^n(16-w)^n}{(n!)^2}e^{-0.00017w}\,dw$$
$$= 0.004775872$$

4

We have that ${}_4p_{37}^{(01)} = {}_2\,p_{37}^{(00)} \times {}_2\,p_{39}^{(01)} + {}_2\,p_{37}^{(01)} \times {}_2\,p_{39}^{(11)}$. For the numbers in the table, this gives

$$0.007857 = 0.992036 \times 0.003960 + 0.003960 \times 0.992054 = 0.007857$$
$$0.007857 = 0.992036 \times 0.003968 + 0.003968 \times 0.992054 = 0.007873$$
$$0.007857 = 0.992036 \times 0.003964 + 0.003964 \times 0.990054 = 0.007857$$

So the second actuary's calculations cannot be right. Furthermore, since $\mu_x^{02} < \mu_x^{12}$ for all x, we should have ${}_4p_{37}^{02} < {}_4\,p_{37}^{12}$, which rules out the first actuary's calculations. This means that only Actuary III's calculations might be correct. [Indeed these are the correct values.]

8.6 Premiums

5

The rate of exit of state 0 is $0.0004+0.000003(t+42) = 0.000526+0.000003t$, so ${}_tp_{42}^{(00)} = e^{-\int_0^t 0.000526+0.000003t\,dt} = e^{-0.000526-0.0000015t^2}$.

Premiums are payable while healthy. If the rate of premium is P, then the expected present value of the premium paid is

$$P\int_0^5 e^{-0.03t}e^{-0.000526t-0.0000015t^2}\,dt$$
$$= P\int_0^5 e^{-0.030526t-0.0000015t^2}\,dt$$
$$= P\int_0^5 e^{-0.0000015(t+10175.333333)^2+0.0000015\times 10175.333333^2}\,dt$$
$$= Pe^{0.0000015\times 10175.333333^2}\sqrt{\frac{\pi}{0.0000015}}\left(\Phi\left(10180.333333\sqrt{0.000003}\right) - \Phi\left(10175.333333\sqrt{0.000003}\right)\right)$$
$$= 4.637064P$$

On the other hand, the expected death benefits for lives that are not critically ill first are:

$$
\begin{aligned}
&100000\int_0^5 e^{-0.03t}e^{-0.000526t-0.0000015t^2}(0.0001+0.000001(t+42))\,dt\\
&=100000\int_0^5 e^{-0.030526t-0.0000015t^2}(0.000142+0.000001t)\,dt\\
&=100000\int_0^5 e^{-0.030526t-0.0000015t^2}(0.000001(t+\frac{0.030526}{0.000003})+0.000142-0.01017533333333)\,dt\\
&=0.1e^{\frac{0.030526^2}{0.000006}}\int_0^5\left(t+\frac{0.030526}{0.000003}\right)e^{-0.0000015\left(t+\frac{0.030526}{0.000003}\right)^2}\,dt-1003.333333333333\int_0^5 e^{-0.030526t-0.0000015t^2}\,dt\\
&=0.1e^{\frac{0.030526^2}{0.000006}}\left[-\frac{e^{-0.0000015\left(t+\frac{0.030526}{0.000003}\right)^2}}{0.000003}\right]_0^5-1003.333333333333\times 4.637064\\
&=66.97587
\end{aligned}
$$

If a life becomes critically ill at age x, the probability that it survives for t years is $e^{-\int_0^t 0.02\,dt}=e^{-0.02t}$. The expected value of the benefits to such a life is therefore given by

$$
90000\int_0^{47-x} e^{-0.03(x+t-42)}e^{-0.02t}\,dt=90000e^{-0.03(x-42)}\int_0^{47-x} e^{-0.05t}\,dt=1800000e^{-0.03(x-42)}(1-e^{-0.05(47-x)})
$$

The expected value of death benefits to such an individual is

$$
100000\int_0^{47-x} 0.02e^{-0.03(x+t-42)}e^{-0.02t}\,dt=2000e^{-0.03(x-42)}\int_0^{47-x} e^{-0.05t}\,dt=4000e^{-0.03(x-42)}(1-e^{-0.05(47-x)})
$$

So the total expected benefits paid to individuals who become disabled are

$$
\begin{aligned}
&1840000\int_0^5 e^{-0.000526s-0.0000015s^2}(0.0003+0.000002(s+42))e^{-0.03(5-s)}(1-e^{-0.05(5-s)})\,ds\\
&=1.84\int_0^5(384+2s)e^{-0.15+0.029474s-0.0000015s^2}(1-e^{-0.05(5-s)})\,ds\\
&=1.84\int_0^5(384+2s)(e^{-0.15+0.029474s-0.0000015s^2}-e^{-0.4+0.079474s-0.0000015s^2})\,ds
\end{aligned}
$$

$$
\begin{aligned}
0.15-0.029474s+0.0000015s^2&=0.0000015(s^2-19649.333333s+100000)\\
&=0.0000015((s-9824.666667)^2-96424075)\\
0.4-0.079474s+0.0000015s^2&=0.0000015(s^2-52982.666667s+266666.666667)\\
&=0.0000015((s-26491.333333)^2-701524075)
\end{aligned}
$$

So the expected benefits are

$$
\begin{aligned}
&1.84\int_0^5 (384+2s)\left(e^{144.6361}e^{-\frac{0.000003(s-9824.666667)^2}{2}} - e^{1052.286}e^{-\frac{0.000003(s-26491.333333)^2}{2}}\right) ds \\
&= 1.84e^{144.6361}\int_0^5 (20033.3333333 + 2(s-9824.67))e^{-\frac{0.000003(s-9824.67)^2}{2}}\,ds - 1.84e^{1052.286}\int_0^5 (53366.67+2(s-26491.33))e^{-\frac{0.000003(s}{}} \\
&= 1.84e^{144.6361}\left(20033.33 \times \sqrt{\frac{2\pi}{0.000003}}\left(\Phi(-9819.67\times\sqrt{0.000003}) - \Phi(-9824.67\times\sqrt{0.000003})\right) - 2\left[\frac{e^{-\frac{0.000003(s-9824.67)^2}{2}}}{0.000003}\right]_0^5\right) \\
&\quad - 1.84e^{1052.286}\left(53366.67 \times \sqrt{\frac{2\pi}{0.000003}}\left(\Phi(-26491.33\times\sqrt{0.000003}) - \Phi(-26496.33\times\sqrt{0.000003})\right) - 2\left[\frac{e^{-\frac{0.000003(s-26491.33)^2}{2}}}{0.000003}\right]_0^5\right. \\
&= 371.8703
\end{aligned}
$$

The total expected benefit is therefore

$$371.8703 + 66.97587 = 438.8461$$

The premium is therefore

$$\frac{438.8461}{4.637572} = \$94.64$$

6

$$
0.000001\begin{pmatrix}
-\left(1843x+38x^2+\frac{x^3}{3}\right) & 374x+x^2 & 1469x+37x^2+\frac{x^3}{3} \\
67x+0.5x^2 & -(341x+1.5x^2) & 274x+x^2 \\
0 & 0 & 0
\end{pmatrix}
$$

We calculate the probability that the life is in each state at the end of each year:

t	${}_tp_{37}^{00}$	${}_tp_{37}^{01}$	${}_tp_{37}^{02}$
0	1	0	0
1	0.99812	0.000375	0.001505
2	0.99617	0.000750	0.003083
3	0.99414	0.001127	0.004736
4	0.99203	0.001505	0.006464
5	0.98985	0.001884	0.008271
6	0.98758	0.002263	0.010156
7	0.98523	0.002644	0.012123
8	0.98280	0.003025	0.014171
9	0.98029	0.003407	0.016303
10	0.97769	0.003790	0.018519

EPV of death benefits is

$$200000\big(0.00150(1.06)^{-1} + 0.00157(1.06)^{-2} + 0.00165(1.06)^{-3} + 0.00172(1.06)^{-4} + 0.00180(1.06)^{-5} + 0.00188(1.06)^{-6} + 0.00197(1.06)^{-7} + 0.00205(1.06)^{-8} + 0.00213(1.06)^{-9} + 0.00222(1.06)^{-10}\big) = 2670.49$$

EPV of disability benefits is

$$80000(0.000375(1.06)^{-1} + 0.000750(1.06)^{-2} + 0.001127(1.06)^{-3} + 0.001505(1.06)^{-4} + 0.001884(1.06)^{-5} + 0.002263(1.06)^{-6} + 0.002644(1.06)^{-7} + 0.003025(1.06)^{-8} + 0.003407(1.06)^{-9} + 0.003790(1.06)^{-10}) = 1116.21$$

EPV of all benefits is $2670.49 + 1116.21 = 3786.70$

EPV of unit premiums is $1 + 0.99812(1.06)^{-1} + 0.99617(1.06)^{-2} + 0.99414(1.06)^{-3} + 0.99203(1.06)^{-4} + 0.98985(1.06)^{-5} + 0.98758(1.06)^{-6} + 0.98523(1.06)^{-7} + 0.98280(1.06)^{-8} + 0.98029(1.06)^{-9} = 7.736653311$

So annual premium is

$$\frac{3786.70}{7.736653311} = \$489.45$$

7

If the rate of premium is P, the EPV of total premiums received is

$$P\overline{a}00_{34:\overline{10}|} = P\left(\overline{a}^{00}_{34} -_{10} p^{00}_{34} e^{-10\delta}\overline{a}^{00}_{44} -_{10} p^{00}_{34} e^{-10\delta}\overline{a}^{10}_{44}\right) = (22.07 - 0.934e^{-0.3}\times 19.30 - 0.022e^{-0.3}\times 0.11)P = 8.71407P$$

The total EPV of benefits are

$$\begin{aligned}
&80000\overline{a}^{01}_{34:\overline{10}|} + 280000\overline{A}^{02}_{34:\overline{10}|} \\
&= 80000\left(\overline{a}^{01}_{34} -_{10} p^{00}_{34}\overline{a}^{01}_{44} -_{10} p^{01}_{34}\overline{a}^{11}_{44}\right) + 280000\left(\overline{A}^{02}_{34} -_{10} p^{00}_{34} e^{-0.3}\overline{A}^{02}_{44} -_{10} p^{01}_{34} e^{-0.3}\overline{A}^{12}_{44}\right) \\
&= 80000(0.64 - 0.934e^{-0.3} \times 0.43 - 0.022e^{-0.3} \times 17.32) + 280000(0.14 - 0.934e^{-0.3} \times 0.19 - 0.022e^{-0.3} \times 0.21) \\
&= 4815.30 + 1431.31 \\
&= 6246.61
\end{aligned}$$

The premium is therefore $\frac{6246.61}{8.71407} = \$716.84$.

8

We calculate

$$\begin{aligned}
\overline{a}_{x:\overline{5}|} &= \int_0^5 e^{-0.03t}(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})\,dt \\
&= \int_0^5 0.2113249e^{-0.036732051t}\,dt + \int_0^5 0.7886751e^{-0.033267949t})\,dt \\
&= \frac{0.2113249}{0.036732051}(1 - e^{-0.036732051\times 5}) + \frac{0.7886751}{0.033267949}(1 - e^{-0.033267949\times 5}) \\
&= 4.598130
\end{aligned}$$

The EPV of the benefits to lives who die accidentally from State 0 are given by

$$
\begin{aligned}
&200000\int_0^5 0.002(0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=400\int_0^5 (0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=400\times 4.598130\\
&=1839.25
\end{aligned}
$$

The EPV of the benefits to lives who die otherwise from State 0 are given by

$$
\begin{aligned}
&100000\int_0^5 0.001(0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=100\int_0^5 (0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=100\times 4.598130\\
&=459.81
\end{aligned}
$$

The EPV of the benefits to lives who die accidentally from State 1 are given by

$$
\begin{aligned}
&200000\int_0^5 0.001(0.2886752e^{-0.003267949t}-0.2886752e^{-0.006732051t})e^{-0.03t}\,dt\\
&=200\int_0^5 0.2886752e^{-0.033267949t}-0.2886752e^{-0.036732051t})\,dt\\
&=\frac{57.73504}{0.033267949}(1-e^{-5\times 0.033267949})-\frac{57.73504}{0.036732051}(1-e^{-5\times 0.036732051})\\
&=2.226627
\end{aligned}
$$

The EPV of the benefits to lives who die otherwise from State 1 are given by

$$
\begin{aligned}
&100000\int_0^5 0.003(0.2886752e^{-0.003267949t}-0.2886752e^{-0.006732051t})e^{-0.03t}\,dt\\
&=300\int_0^5 0.2886752e^{-0.033267949t}-0.2886752e^{-0.036732051t})\,dt\\
&=3.339940
\end{aligned}
$$

The total EPV of benefits is therefore $1839.25+459.81+2.23+3.34=\$2,304.63$.

The annual rate of premium is $\frac{2304.63}{4.598130} = \$501.21$.
9
(a) In this case, additional premiums are due for the first three months of each period of sickness. We have

$$\begin{aligned} {}_tp_x^{\overline{11}} &= e^{-0.006t} \\ \bar{a}_{x:\overline{s}|}^{\overline{11}} &= \int_0^s e^{-0.006t} e^{-0.03t} dt \\ &= \frac{1-e^{-0.036s}}{0.036} \end{aligned}$$

The EPV of premiums at rate 1 due during the first 3 months of sickness is given by

$$\begin{aligned} &\int_0^{4.75} e^{-\delta t} {}_tp_x^{00} \mu_{x+t}^{01} \frac{1-e^{-0.036\times 0.25}}{0.036}\, dt + \int_{4.75}^5 e^{-\delta t} {}_tp_{x+t}^{00} \mu_{x+t}^{01} \frac{1-e^{-0.036(5-t)}}{0.036} \\ =&\frac{1-e^{-0.009}}{0.036} \int_0^{4.75} e^{-0.03t}(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})0.001\, dt \\ &+ \int_{4.75}^5 e^{-0.03t}(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})0.001 \frac{1-e^{-0.036(5-t)}}{0.036}\, dt \\ =&\left(\frac{1-e^{-0.009}}{36}\right)\left(\frac{0.2113249(1-e^{-0.036732051\times 0.475})}{0.036732051} + \frac{0.7886751(1-e^{-0.033267949\times 0.475})}{0.033267949}\right) \\ &+ \frac{1}{36}\int_{4.75}^5 (0.2113249e^{-0.0367332051t} + 0.7886751e^{-0.033267949t})\, dt \\ &- \frac{1}{36}\int_{4.75}^5 (0.2113249e^{-0.036732051t} + 0.7886751e^{-0.033267949t})e^{-0.036(5-t)}\, dt \\ =&0.0005866816 + \frac{1}{36}\left(\frac{0.2113249(e^{-0.036732051\times 4.75} - e^{-0.036732051\times 5})}{0.036732051} + \frac{0.7886751(e^{-0.033267949\times 4.75} - e^{-0.033267949\times 5})}{0.033267949}\right) \\ &- \frac{e^{-0.18}}{36}\int_{4.75}^5 (0.2113249e^{-0.000732051t} + 0.7886751e^{0.002732051t})\, dt \\ =&0.0005866816 + 0.005883893 \\ &- \frac{e^{-0.18}}{36}\left(\frac{0.2113249(e^{-0.000732051\times 4.75} - e^{-0.000732051\times 5})}{0.000732051} + \frac{0.7886751(e^{0.002732051\times 5} - e^{0.002732051\times 4.75})}{0.002732051}\right) \\ =&0.0005866816 + 0.005883893 - 0.005857458 \\ =&0.0006131172 \end{aligned}$$

The new annuity value is therefore $4.598130 + 0.0006131172 = 4.598761$, so the annual rate of premium is $\frac{2304.63}{4.598761} = \$501.14$.

(b) This situation is more complicated. One natural approach would be to add an additional state to represent the off-period (i.e. a new state representing healthy lives who had just recovered from disability, and would therefore have no waiting time, or a reduced waiting period if they become disabled). The trouble with this approach is that transitions from this off-period state are not Markovian — lives transition to healthy once they have been in this state for 6 months.

As a simplifying approximation, we will assume that all periods of disability last at least 3 months and so use up all the waiting time. Since the time spent in the disabled state is exponentially distributed with

$\lambda = 0.002 + 0.001 + 0.003 = 0.006$, the probability that a period of disability lasts for less than 3 months is $1 - e^{-0.0015} = 0.001498876$, so this approximation should not have a big effect on our estimates.

Suppose the life is healthy at time t. We want to calculate the distribution of the time since they were last disabled (if ever). The density of the time they last stopped being disabled is given by

$$f(s) = \frac{{}_sp_x^{01}\mu_{x+s}^{10}{}_{t-s}p_{x+s}^{\overline{00}}}{{}_tp_x^{00}} = 0.002e^{-0.004(t-s)}\frac{0.2886752e^{-0.003267949s} - 0.2886752e^{-0.006732051s}}{0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t}}$$

The probability $p_d(t)$ that the individual was disabled at some point during the preceding 6 months is therefore (for $t > 0.5$)

$$\begin{aligned}
p_d(t) &= \int_{t-0.5}^{t} f(s)\,ds \\
&= \frac{0.002e^{-0.004t}}{0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t}}\int_{t-0.5}^{t} e^{0.004s}\left(0.2886752e^{-0.003267949s} - 0.2886752e^{-0.006732051s}\right)ds \\
&= \frac{0.002 \times 0.2886752e^{-0.004t}}{0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t}}\int_{t-0.5}^{t}\left(e^{0.000732051s} - e^{-0.002732051s}\right)ds \\
&= \frac{0.002 \times 0.2886752}{0.2113249e^{-0.002732051t} + 0.7886751e^{0.000732051t}}\int_{t-0.5}^{t}\left(e^{0.000732051s} - e^{-0.002732051s}\right)ds \\
&= \frac{0.002 \times 0.2886752}{0.2113249e^{-0.002732051t} + 0.7886751e^{0.000732051t}}\left(\frac{e^{0.000732051t} - e^{0.000732051(t-0.5)}}{0.000732051} - \frac{e^{-0.002732051(t-0.5)} - e^{-0.002732051t}}{0.002732051}\right) \\
&= \frac{0.002 \times 0.2886752}{0.2113249e^{-0.002732051t} + 0.7886751e^{0.000732051t}}\left(e^{0.000732051t}\left(\frac{1 - e^{-0.000366026}}{0.000732051}\right) - e^{-0.002732051t}\left(\frac{e^{0.001366026} - 1}{0.002732051}\right)\right) \\
&= 0.0005773504\frac{0.4999092e^{0.000732051t} - 0.5003418e^{-0.002732051t}}{0.2113249e^{-0.002732051t} + 0.7886751e^{0.000732051t}} \\
&= 0.0005773504\frac{0.4999092e^{0.000732051t} - 0.5003418e^{-0.002732051t}}{0.2113249e^{-0.002732051t} + 0.7886751e^{0.000732051t}} \\
&= 0.0005773504\frac{0.4999092e^{-0.003267949t} - 0.5003418e^{-0.006732051t}}{{}_tp_x^{00}}
\end{aligned}$$

For $t < 0.5$ we have

$$\begin{aligned}
p_d(t) &= \int_0^t f(s)\,ds \\
&= \frac{0.002e^{-0.004t}}{{}_tp_x^{00}} \int_0^t e^{0.004s}\left(0.2886752e^{-0.003267949s} - 0.2886752e^{-0.006732051s}\right) ds \\
&= \frac{0.002 \times 0.2886752e^{-0.004t}}{{}_tp_x^{00}} \int_0^t \left(e^{0.000732051s} - e^{-0.002732051s}\right) ds \\
&= \frac{0.0005773504e^{-0.004t}}{{}_tp_x^{00}} \int_0^t \left(e^{0.000732051s} - e^{-0.002732051s}\right) ds \\
&= \frac{0.0005773504e^{-0.004t}}{{}_tp_x^{00}} \left(\frac{e^{0.000732051t} - 1}{0.000732051} - \frac{1 - e^{-0.002732051t}}{0.002732051}\right) \\
&= \frac{0.0005773504}{{}_tp_x^{00}} \left(\frac{e^{-0.003267949t} - e^{-0.004t}}{0.000732051} - \frac{e^{-0.004t} - e^{-0.006732051t}}{0.002732051}\right) \\
&= \frac{0.7886751e^{-0.003267949t} + 0.2113249e^{-0.006732051t} - e^{-0.004t}}{{}_tp_x^{00}} \\
&= 1 - \frac{e^{-0.004t}}{{}_tp_x^{00}}
\end{aligned}$$

Now we will use the same method as in (a), except that whenever the life becomes sick, the waiting period is 0 with probability $p_d(t)$ and 0.25 with probability $1 - p_d(t)$.

We already calculated in (a), that given W, the expected premiums during this waiting period are given by $\overline{a}^{\overline{11}}_{x:\overline{0.25|}} = \frac{1-e^{-0.036\times 0.25)}}{0.036} = \frac{1-e^{-0.009}}{0.036} = 0.2488784$.

As in part (a) we calculate the EPV of premiums during the first 3 months of disability that are a continuation of an earlier period of disability (so this is the difference between the answer and the answer to part (a)):

$$\int_0^{0.5} p_d(t)e^{-\delta t}{}_tp_x^{00}\mu_{x+t}^{01}\frac{1-e^{-0.036\times 0.25}}{0.036}\,dt + \int_{0.5}^{4.75} p_d(t)e^{-\delta t}{}_tp_x^{00}\mu_{x+t}^{01}\frac{1-e^{-0.036\times 0.25}}{0.036}\,dt + \int_{4.75}^{5} p_d(t)e^{-\delta t}{}_tp_x^{00}\mu_{x+t}^{01}\frac{1-e^{-0.036(5-t)}}{0.036}$$

$$=\left(\frac{1-e^{-0.009}}{0.036}\right)\left(\int_0^{0.5} e^{-0.03t}{}_tp_x^{00}0.001\,dt - \int_0^{0.5} e^{-0.03t}e^{-0.004t}0.001\,dt\right)$$

$$+0.0005773504\left(\frac{(1-e^{-0.009})}{0.036}\int_{0.5}^{4.75} e^{-0.03t}(0.4999092e^{-0.003267949t}-0.5003418e^{-0.006732051t})0.001\,dt\right.$$

$$\left.+\int_{4.75}^{5} e^{-0.03t}(0.4999092e^{-0.003267949t}-0.5003418e^{-0.006732051t})0.001\frac{1-e^{-0.036(5-t)}}{0.036}\,dt\right)$$

$$=\left(\frac{1-e^{-0.009}}{36}\right)\left(\frac{0.2113249(1-e^{-0.036732051\times 0.5})}{0.036732051}+\frac{0.7886751(1-e^{-0.033267949\times 0.5})}{0.033267949}\right)$$

$$+0.0005773504\left(\left(\frac{1-e^{-0.009}}{36}\right)\left(\frac{0.4999092(e^{-0.033267949\times 0.5}-e^{-0.033267949\times 4.75})}{0.033267949}-0.5003418\frac{(e^{-0.036732051\times 0.5}-e^{-0.03673}}{0.036732051}\right.\right.$$

$$+\frac{1}{36}\int_{4.75}^{5}(0.4999092e^{-0.033267949t}-0.5003418e^{-0.036732051t})\,dt$$

$$\left.-\frac{1}{36}\int_{4.75}^{5}(0.4999092e^{-0.033267949t}-0.5003418e^{-0.036732051t})e^{-0.036(5-t)}\,dt\right)$$

$$=0.0001233874+0.0005773504\left(3.886234\times 10^{-06}\right.$$

$$+\frac{1}{36}\left(\frac{0.2113249(e^{-0.036732051\times 4.75}-e^{-0.036732051\times 5})}{0.036732051}+\frac{0.7886751(e^{-0.033267949\times 4.75}-e^{-0.033267949\times 5})}{0.033267949}\right)$$

$$\left.-\frac{e^{-0.18}}{36}\left(\frac{0.2113249(e^{-0.000732051\times 4.75}-e^{-0.000732051\times 5})}{0.000732051}+\frac{0.7886751(e^{0.002732051\times 5}-e^{0.002732051\times 4.75})}{0.002732051}\right)\right)$$

$$=0.0001233874+0.0005773504(3.886234\times 10^{-06}+0.005883893-0.005857458)$$

$$=0.0001234049$$

Therefore, under this model, we have $\overline{a}_{x:\overline{5}|} = 4.598761 - 0.0001234049 = 4.598638$. The premium is therefore $\frac{2304.63}{4.598638} = \$501.15$.

8.7 Policy Values and Thiele's Differential Equation

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Thiele's differential equation gives

$$
\begin{aligned}
\frac{d}{dt}{}_tv^{(1)} &= \delta_t v^{(1)} + P^{(1)} - B^{(1)} - \sum_{j\neq 1} \mu_{x+t}^{(1j)}(S^{(1j)} +_t v^{(j)} -_t v^{(1)}) \\
&= 0.03{}_tv^{(1)} - 90000 - \mu_{x+t}^{(12)}(100000 - {}_tv^{(1)}) \\
&= 0.03{}_tv^{(1)} - 90000 - 0.02(100000 - {}_tv^{(1)}) \\
&= 0.05{}_tv^{(1)} - 92000 \\
{}_tv^{(1)} &= 1840000(1 - e^{0.05(t-5)})
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt}{}_tv^{(0)} &= \delta_t v^{(0)} + P^{(0)} - B^{(0)} - \sum_{j\neq 0} \mu_{x+t}^{(0j)}(S^{(0j)} +_t v^{(j)} -_t v^{(0)}) \\
&= 0.03{}_tv^{(0)} + 71.83 - 0 - \mu_{x+t}^{(01)}({}_tv^{(1)} -_t v^{(0)}) - \mu_{x+t}^{(02)}(100000 - {}_tv^{(0)}) \\
&= 0.03{}_tv^{(0)} + 71.83 - (0.0003 + 0.000002(42+t))({}_tv^{(1)} -_t v^{(0)}) - (0.0001 + 0.000001(42+t))(100000 - {}_tv^{(0)}) \\
&= 0.03{}_tv^{(0)} + 71.83 - (0.000384 + 0.000002t)(1840000(1 - e^{0.05(t-5)}) -_t v^{(0)}) - (0.000142 + 0.000001t)(100000 - {}_tv^{(0)}) \\
&= 71.83 - 706.56 - 14.2 - 3.68t(1 - e^{0.05(t-5)}) - 0.1t + (0.03 + 0.0000384 + 0.000142){}_tv^{(0)} + 0.000003t{}_tv^{(0)} + 70.656e^{0.05(t-5} \\
&= -13.026 - 3.78t + 3.68te^{0.05(t-5)} + 0.0300526{}_tv^{(0)} + 0.000003t{}_tv^{(0)} + 706.56e^{0.05(t-5)}
\end{aligned}
$$

8.9 Multiple Decrement Tables

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We have ${}_tp_x^{(00)} = e^{-0.0023t-0.000005((x+t)^2-x^2)}$, so we have

$$
\begin{aligned}
\overline{a}_{36:\overline{10}|}^{(00)} &= \int_0^{10} e^{-0.04t} e^{-0.0023t-0.000005((36+t)^2-36^2)}\,dt \\
&= \int_0^{10} e^{-0.04266t-0.000005t^2}\,dt \\
&= e^{\frac{4266}{200000}} \int_0^{10} e^{-\frac{(t+4266)^2}{200000}}\,dt \\
&= 100\sqrt{20\pi}e^{\frac{4266^2}{20000}} \left(\Phi\left(\frac{42.76}{\sqrt{10}}\right) - \Phi\left(\frac{42.66}{\sqrt{10}}\right)\right) = 8.139325
\end{aligned}
$$

$$\begin{aligned}
\overline{A}^{(01)}_{36:\overline{10}|} &= \int_0^{10} (0.00102 + 0.00002t)e^{-0.04t}e^{-0.0023t-0.000005((36+t)^2-36^2)}\,dt \\
&= e^{\frac{4266^2}{200000}} \int_0^{10} 0.00002(t+51)e^{-\frac{(t+4266)^2}{200000}}\,dt \\
&= 0.00002e^{\frac{4266^2}{200000}} \left(\int_0^{10} (t+4266)e^{-\frac{(t+4266)^2}{200000}}\,dt - 4215 \int_0^{10} e^{-\frac{(t+4266)^2}{200000}}\,dt \right) \\
&= 0.00002e^{\frac{4266^2}{200000}} \left(\left[-100000e^{-\frac{(t+4266)^2}{200000}} \right]_0^{10} - 421500\sqrt{20\pi} \left(\Phi\left(\frac{42.76}{\sqrt{10}}\right) - \Phi\left(\frac{42.66}{\sqrt{10}}\right) \right) \right) \\
&= 0.009058283
\end{aligned}$$

So the annual rate of premium is

$$\frac{300000 \times 0.01093749}{8.007432} = \$333.87$$

(b) For policies which do not lapse, we only need to consider the mortality rate. This gives

$$\begin{aligned}
\overline{a}^{(00)}_{36+t:\overline{10-t}|} &= \int_0^{10-t} e^{-0.04s}e^{-0.0003s-0.000005((36+t+s)^2-(36+t)^2)}\,ds \\
&= \int_0^{10-t} e^{-(0.04102+0.00001t)s-0.000005s^2}\,ds \\
&= e^{\frac{4102^2}{200000}} \int_0^{10-t} e^{-\frac{(s+t+4102)^2}{200000}}\,ds \\
&= 100\sqrt{20\pi}e^{\frac{(t+4102)^2}{200000}} \left(\Phi\left(\frac{41.12}{\sqrt{10}}\right) - \Phi\left(\frac{t+4102}{100\sqrt{10}}\right) \right)
\end{aligned}$$

and

$$\overline{A}^{(00)}_{36+t:\overline{10-t}|} = 1 - 0.04\overline{a}^{(00)}_{36+t:\overline{10-t}|}$$

So the policy value is

$$\begin{aligned}
{}_tV &= 300000(1 - 0.04\overline{a}^{(00)}_{36+t:\overline{10-t}|} - {}_{10-t}\,p_{36+t}e^{-0.04(10-t)}) - P\overline{a}^{(00)}_{36+t:\overline{10-t}|} \\
&= 300000 - (12000+P)\overline{a}^{(00)}_{36+t:\overline{10-t}|} - 300000\,{}_{10-t}p_{36+t}e^{-0.04(10-t)} \\
&= 300000 - (12000+P)100\sqrt{20\pi}e^{\frac{(t+4102)^2}{200000}} \left(\Phi\left(\frac{41.12}{\sqrt{10}}\right) - \Phi\left(\frac{t+4102}{100\sqrt{10}}\right) \right) - 300000\,{}_{10-t}p_{36+t}e^{-0.04(10-t)} \\
&= a(t) + b(t)P
\end{aligned}$$

for some functions $a(t)$ and $b(t)$.
From (a), the expected death benefit of the policy is $300000 \times 0.01093749 = \$3281.25$.
The expected surrender benefit for the policy is

$$\begin{aligned}
\frac{1}{2}\int_0^{10} e^{-0.04t}{}_tp_{36}\mu_{36+t}^{(01)}{}_tV\,dt &= \frac{1}{2}\int_0^{10} e^{-0.04t}e^{-(0.0023t+0.000005(t^2+72t))}(0.002-0.00001(36+t))_tV\,dt \\
&= \frac{1}{2}\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))_tV\,dt \\
&= \frac{1}{2}\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))(a(t)+b(t)P)\,dt \\
&= \frac{1}{2}\left(\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))a(t)\,dt\right. \\
&\qquad \left.+\int_0^{10} e^{-0.000005(t^2+8532t)}(0.00001(164-t))b(t)P\,dt\right) \\
&= \frac{1}{2}(46.51242-0.08175562P)
\end{aligned}$$

Using numerical integration. We note that this requires the policy value to always be positive.
We therefore have

$$\begin{aligned}
8.007432P &= 3281.247+23.25621-0.04087781P \\
7.966554P &= 3304.50321 \\
P &= \$414.80
\end{aligned}$$

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We have

$$ {}_tp_x^{(00)} = e^{-\int_0^t 0.0023+0.00001(x+s)\,ds} = e^{-((0.0023+0.00001x)t+0.000005t^2)} $$

In particular,

$$ {}_tp_{29}^{(00)} = e^{-(0.00259t+0.000005t^2)} $$

We therefore calculate

$$\begin{aligned}
\overline{a}_{29:\overline{10}|}^{(00)} &= \int_0^{10} e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\,dt \\
&= \int_0^{10} e^{-(0.05259t+0.000005t^2)}\,dt \\
&= e^{\frac{5259^2}{200000}}\int_0^{10} e^{-\frac{(t+5259)^2}{200000}}\,dt \\
&= e^{\frac{5259^2}{200000}}100\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right) \\
&= 7.775573
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
\overline{A}^{(01)}_{29:\overline{10}|} &= \int_0^{10} 0.0003e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\,dt \\
&= 0.0003\int_0^{10} e^{-(0.05259t+0.000005t^2)}\,dt \\
&= 0.0003e^{\frac{5259^2}{200000}}\int_0^{10} e^{-\frac{(t+5259)^2}{200000}}\,dt \\
&= 0.0003e^{\frac{5259^2}{200000}}100\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right) \\
&= 0.002332672
\end{aligned}$$

and

$$\begin{aligned}
\overline{A}^{(02)}_{29:\overline{10}|} &= \int_0^{10} 0.00002(29+t)e^{-0.05t}e^{-(0.00259t+0.000005t^2)}\,dt \\
&= \int_0^{10} 0.00002(29+t)e^{-(0.05259t+0.000005t^2)}\,dt \\
&= 0.00002e^{\frac{5259^2}{200000}}\int_0^{10}(t+5259-5230)e^{-\frac{(t+5259)^2}{200000}}\,dt \\
&= 0.00002e^{\frac{5259^2}{200000}}\left(100000\left(e^{-\frac{(5259)^2}{200000}}-e^{-\frac{(5269)^2}{200000}}\right)-523000\sqrt{20\pi}\left(\Phi\left(\frac{52.69}{\sqrt{10}}\right)-\Phi\left(\frac{52.59}{\sqrt{10}}\right)\right)\right) \\
&= 0.005219487
\end{aligned}$$

The EPV of benefits is therefore

$$100000\times 0.005219487+200000\times 0.002332672=988.48$$

and the premium is

$$\frac{988.48}{7.775573}=\$127.13$$

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$$\frac{56.94+56.61}{9878.44}=0.01149473$$

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We calculate

$$A^{(02)}{}^{\,1}_{40:\overline{5}|} = \frac{0.30(1.03)^{-1} + 0.29(1.03)^{-2} + 0.28(1.03)^{-3} + 0.27(1.03)^{-4} + 0.27(1.03)^{-5}}{10000} = 0.000129365$$
$$A^{(03)}{}^{\,1}_{40:\overline{5}|} = \frac{1.62(1.03)^{-1} + 1.70(1.03)^{-2} + 1.78(1.03)^{-3} + 1.89(1.03)^{-4} + 1.98(1.03)^{-5}}{10000} = 0.0008191387$$
$$\ddot{a}_{40:\overline{5}|} = 1 + 0.993908(1.03)^{-1} + 0.987844(1.03)^{-2} + 0.981806(1.03)^{-3} + 0.975795(1.03)^{-4} = 4.66157$$

So the net premium is $\frac{200000\times0.000129365+100000\times0.0008191387}{4.66157} = \$23.12$

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(a) Using UDD in the multiple decrement table, the probability that the policy is still in force at age 46 years 3 months is $\frac{\frac{1}{4}9579.02+\frac{3}{4}0.963844}{\frac{1}{3}10000+\frac{2}{3}9939.08} = 0.9662829$.

The probability that the life then dies in an accident before age 47 is $\frac{3}{4}\times\frac{0.25}{\frac{1}{4}\times9579.02+\frac{3}{4}\times9638.44} = 0.00001948338$.

The probability that the life dies in an accident between ages 46 and 3 months and 47 is therefore $0.9662829\times 0.00001948338 = 0.00001882646$.

The probability that the life survives to age 47 is $\frac{9579.02}{\frac{1}{3}10000+\frac{2}{3}9939.08} = 0.9618082$.

The probability that a life aged 47 dies in an accident before age 47 and 5 months is $\frac{5}{12}\times\frac{0.24}{9579.02}$. The probability that the life dies in an accident between ages 47 and 47 and 5 months is therefore $\frac{5}{12}\times\frac{0.24}{9579.02}\times\frac{9579.02}{\frac{1}{3}10000+\frac{2}{3}9939.08} = 0.00001004078$.

The total probability is therefore
$0.00001882646 + 0.00001004078 = 0.00002886724$.

(b)

Using constant transition intensities, the rate of decrement for a life aged 42–43 is $-\log\left(\frac{9939.08}{10000}\right) = 0.006110632$. The probability that a policy starting at age 42 is still in force at age 42 and 4 months is $e^{-\frac{0.006110632}{3}} = 0.9979652$. The intensity of decrement aged 46–47 is $-\log\left(\frac{9579.02}{9638.44}\right) = 0.00618398$. This gives that $l^{(0)}_{46.25} = 9638.44e^{-0.25\times0.00618398} = 9623.55$. The probability that the policy is still in force at age 46 and 3 months is therefore $\frac{9623.55}{9979.65} = 0.9643174$. The intensity of accidental deaths between ages 46–47 is

$$\frac{0.25}{9638.44\int_0^1 e^{-0.00618398x}\,dx}$$

and the probability of accidental death between ages 46 and 3 months and 47 is therefore

$$\frac{0.25}{9638.44\int_0^1 e^{-0.00618398x}\,dx}\int_0^{0.75} e^{-0.00618398x}\,dx = \frac{0.25}{9638.44}\frac{1-e^{-0.00618398\times0.75}}{1-e^{-0.00618398}} = 0.00001946839$$

The probability of surviving to age 47 is $\frac{9579.02}{9979.65} = 0.9598551$.
The intensity of decrements between ages 47–48 is
$-\log\left(\frac{9519.81}{9579.02}\right) = 0.0062004$.
The intensity of accidental deaths between ages 47–48 is

$$\frac{0.24}{9579.02\int_0^1 e^{-0.0062004x}\,dx}$$

and the probability of accidental death between ages 47 and 47 and 5 months is therefore

$$\frac{0.24}{9579.02\int_0^1 e^{-0.00620040.00618398x}\,dx}\int_0^{\frac{5}{12}} e^{-0.0062004x}\,dx = \frac{0.24}{9579.02}\frac{1-e^{-0.0062004\times\frac{5}{12}}}{1-e^{-0.0062004}} = 0.00001045836$$

The total probability of dying in an accident between ages 46 years 3 months and 47 years 5 months is therefore

$$0.00001946839 \times 0.9643174 + 0.00001045836 \times 0.9598551 = 0.00002881222$$

8.10 Constructing a Multiple Decrement Table

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(a) Under UDD, suppose there are x policies at the start of the year, and during the year, y surrender and z die. Assuming UDD, the number of policies still in force at time t is $x-(y+z)t$, so the rate of dying is $\frac{z}{x-(y+z)t}$. Assume that the deaths in the new table are uniformly distributed over the year. (This is inconsistent with our uniform distribution assumption for policies with surrender, but what the hell.) If the updated mortality table has w deaths from v lives, then the rate of death at time t is $\frac{v}{w-vt}$. With this update, let x_t be the number of policies still in force at time t. We have

$$\frac{dx_t}{dt} = -\left(\frac{y}{x-(y+z)t} + \frac{v}{w-vt}\right)x_t$$

This gives

$$\begin{aligned} x_1 &= xe^{-\int_0^1 \frac{y}{x-(y+z)t}+\frac{v}{w-vt}\,dt} = e^{[y\log(x-(y+z)t)+v\log(w-vt)]_0^1} = e^{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)+\log\left(\frac{w-v}{w}\right)} \\ &= \left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right) \end{aligned}$$

In the new table, we get

$$\begin{aligned} \left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}} &= \left(\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)\right)^{\frac{y'}{y'+z'}} \\ \frac{y'}{y'+z'} &= \frac{y}{y+z}\left(\frac{\log\left(\frac{x-(y+z)}{x}\right)}{\log\left(\left(\frac{x-(y+z)}{x}\right)^{\frac{y}{y+z}}\left(\frac{w-v}{w}\right)\right)}\right) \\ &= \frac{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)}{\frac{y}{y+z}\log\left(\frac{x-(y+z)}{x}\right)+\log\left(\frac{w-v}{w}\right)} \end{aligned}$$

Using this, we can calculate the probabilities on the following slide.

(b) Suppose there are x policies at the start of the year, and during the year, y surrender and z die. Under constant transition probabilities, we have the number of policies still in force at time t is $xe^{\log\left(\frac{x-y-z}{x}\right)t} = x\left(\frac{x-y-z}{x}\right)^t$. If the constant rate of surrender is $\mu^{(01)}$, then

$$y = \int_0^1 \mu^{(01)} x e^{\log\left(\frac{x-y-z}{x}\right)t}\,dt = \mu^{(01)} x \frac{1-\frac{x-y-z}{x}}{\log(x)-\log(x-y-z)} = \mu^{(01)} \frac{y+z}{\log(x)-\log(x-y-z)}$$

This gives $\mu^{(01)} = \frac{y(\log(x)-\log(x-y-z))}{(y+z)}$, so without deaths, the probability of surrender is $e^{-(\log(x)-\log(x-y-z))\frac{y}{(y+z)}} = \left(\frac{x-y-z}{x}\right)^{\frac{y}{y+z}}$. The table is therefore the same as in part (a).

(c) If each independent decrement satisfies UDD, then suppose the probabilities for surrender and death as only decrements are p and q respectively, then the rate of surrender is $\frac{p}{1-pt}$ and the rate of death is $\frac{q}{1-qt}$.

Now in a model with two decrements, the total rate of decrement is $\frac{p}{1-pt} + \frac{q}{1-qt}$, so the total probability of no decrement in the year is

$$e^{-\int_0^1 \frac{p}{1-pt}+\frac{q}{1-qt}\,dt} = e^{[\log(1-pt)+\log(1-qt)]_0^1} = (1-p)(1-q)$$

Similarly, the probability of no decrement before time t is $(1-pt)(1-qt)$. The probability of surrender is

$$\int_0^1 (1-pt)(1-qt)\frac{p}{1-pt}\,dt = \int_0^1 p(1-qt)\,dt = p\left(1-\frac{q}{2}\right)$$

Similarly, the probability of death is $q\left(1-\frac{p}{2}\right)$.

If we are given the probabilities of surrender and death in the multiple decrement model are a and b respectively, then we have to solve

$$\begin{aligned}
p\left(1-\frac{q}{2}\right) &= a\\
q\left(1-\frac{p}{2}\right) &= b\\
p-q &= a-b\\
p-\frac{p^2}{2} &= a-\frac{p}{2}(a-b)\\
p^2+(b-a-2)p+2a &= 0\\
p &= \frac{a+2-b\pm\sqrt{(a+2-b)^2-8a}}{2}\\
&= \frac{a+2-b\pm\sqrt{a^2+b^2+4-2ab-4a-4b)}}{2}\\
q &= \frac{b+2-a\pm\sqrt{a^2+b^2+4-2ab-4a-4b)}}{2}
\end{aligned}$$

x	l_x	d_x
40	10000.00	59.01
41	9940.99	59.02
42	9881.98	59.03
43	9822.95	59.04
44	9763.91	59.06
45	9704.85	59.07
46	9645.78	59.08
47	9586.69	59.11

9.2 Joint Life and Last Survivor Benefits

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Advantages	Disadvantages
Annuity value does not depend on time of death	Value of benefit varies with time of death

9.4 Independent Future Lifetimes

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$$\begin{aligned}\ddot{a}_{63,62:\overline{10}|} &= 1 + 0.999830\frac{9996.47}{9998.30}(1.07)^{-1} + 0.999647\frac{999449}{9998.30}(1.07)^{-2} + 0.999449\frac{9992.35}{9998.30}(1.07)^{-3} + 0.999235\frac{9990.03}{9998.30}(1.07)^{-4} \\ &\quad + 0.999003\frac{9987.53}{9998.30}(1.07)^{-5} + 0.998753\frac{9984.83}{9998.30}(1.07)^{-6} + 0.998483\frac{9981.91}{9998.30}(1.07)^{-7} + 0.998191\frac{9978.76}{9998.30}(1.07)^{-8} \\ &\quad + 0.997876\frac{9975.34}{9998.30}(1.07)^{-9} \\ &= 7.502352\end{aligned}$$

$$
\begin{aligned}
\ddot{a}_{\overline{63,62}:\overline{10|}} &= 0.0000170 \times \frac{1.83}{9998.30}(1.07)^{-1} + \left(0.0000353 \times \frac{1.83+1.98}{9998.30} - 0.0000170 \times \frac{1.83}{9998.30}\right)(1.07)^{-2} \\
&+ \left(0.0000551 \times \frac{1.83+1.98+2.14}{9998.30} - 0.0000353 \times \frac{1.83+1.98}{9998.30}\right)(1.07)^{-3} \\
&+ \left(0.0000765 \times \frac{1.83+1.98+2.14+2.31}{9998.30} - 0.0000551 \times \frac{1.83+1.98+2.14}{9998.30}\right)(1.07)^{-4} \\
&+ \left(0.0000996 \times \frac{1.83+1.98+2.14+2.31+2.50}{9998.30} - 0.0000765 \times \frac{1.83+1.98+2.14+2.31}{9998.30}\right)(1.07)^{-5} \\
&+ \left(0.0001246 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70}{9998.30} - 0.0000996 \times \frac{1.83+1.98+2.14+2.31+2.50}{9998.30}\right)(1.07)^{-6} \\
&+ \left(0.0001516 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92}{9998.30} \right.\\
&\qquad \left. -0.0001246 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70}{9998.30}\right)(1.07)^{-7} \\
&+ \left(0.0001808 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16}{9998.30} \right.\\
&\qquad \left. -0.0001516 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92}{9998.30}\right)(1.07)^{-8} \\
&+ \left(0.0002124 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16+3.41}{9998.30} \right.\\
&\qquad \left. -0.0001808 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16}{9998.30}\right)(1.07)^{-9} \\
&+ \left(0.0002465 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16+3.41+3.69}{9998.30} \right.\\
&\qquad \left. -0.0002124 \times \frac{1.83+1.98+2.14+2.31+2.50+2.70+2.92+3.16+3.41}{9998.30}\right)(1.07)^{-10} \\
&= 0.0000003531257
\end{aligned}
$$

So the premium is $\frac{2000000 \times 0.0000003531257}{7.502352} = \$0.09$.

20

If the Husband is dead and the wife is alive at the end of the policy with probability 0.997567×0.001251, then the wife receives a reversionary annuity with value $30000 \times 13.89755 = 416926.5$. The expected present value of payments after the end of the policy is therefore $30000 \times 13.89755 \times 0.997567 \times 0.001251(1.05)^{-7} = 369.7718$. Payments during the term of the policy are received if and only if the husband is dead and the wife is alive. The expected present value of payments received during the term of the policy is therefore

$$
\begin{aligned}
30000 \, &\left(0.000135 \times 0.999736(1.05)^{-1} + 0.000283 \times 0.999448(1.05)^{-2} + 0.000444 \times 0.999134(1.05)^{-3} \right.\\
&\left. +0.000620 \times 0.998791(1.05)^{-4} + 0.000812 \times 0.998419(1.05)^{-5} + 0.001022 \times 0.998012(1.05)^{-6}\right) = 80.22267
\end{aligned}
$$

So the total expected benefit is $369.7718 + 80.22267 = 449.9945$.

For the premiums, we have

$$\ddot{a}_{53,64:\overline{7}|} = 1+0.999865\times0.999736(1.05)^{-1}+0.999717\times0.999448(1.05)^{-2}+0.999555\times0.999134(1.05)^{-3}+0.999380\times0.998791(1.05)^{-4}+0$$

So the net annual premium is $\frac{80.22267}{6.067799} = \$13.22$.

21

The probabilities of each being dead after n years are:

Years	P(Husband Dead)	P(Wife Dead)	P(Survivor)	EPV benefit
0	0.000000	0.000000	1.0000000	1.0000000
1	0.057684	0.000103	0.9999941	0.9433906
2	0.120792	0.000213	0.9999743	0.8899735
3	0.189153	0.000331	0.9999374	0.8395667
4	0.262348	0.000457	0.9998801	0.7919987
5	0.339656	0.000592	0.9997989	0.7471079
6	0.420004	0.000736	0.9996909	0.7047426
7	0.501937	0.000891	0.9995528	0.6647597
8	0.583624	0.001057	0.9993831	0.6270253
9	0.662908	0.001236	0.9991806	0.5914135
10	0.737429	0.001428	0.9989470	0.5578068
11	0.804821	0.001635	0.9986841	0.5260943
12	0.862981	0.001858	0.9983966	0.4961725
13	0.910384	0.002098	0.9980900	0.4679435
14	0.946379	0.002357	0.9977694	0.4413144
15	0.971384	0.002637	0.9974385	0.4161962
16	0.986877	0.002939	0.9970996	0.3925045
17	0.995126	0.003265	0.9967509	0.3701578
18	0.998683	0.003617	0.9963878	0.3490783
19	0.999799	0.003998	0.9960028	0.3291919
total				12.14644

The EPV of benefits after 20 years is $0.9955900(1.06)^{-20} \times 16.1807 = 5.022969$, so total EPV is $12.14644 + 5.022969 = 17.16941$.

Net premium is $45000 \times 17.16941 = \$772,623.45$.

22

Year	P(Husband Dies)	P(Wife Dies)	P(Both die)	P(One dies)
1	0.000180	0.001651	0.00000029718	0.001830994
2	0.000193	0.001785	0.00000035343	0.001977993
3	0.000208	0.001929	0.000000401232	0.002136992
4	0.000223	0.002085	0.000000464955	0.002307991
5	0.000240	0.002254	0.00000054096	0.002493989
6	0.000258	0.002435	0.00000062823	0.002692987
7	0.000278	0.002631	0.000000731418	0.002908985

Annual EPV from one dying: 0.0138451

Annual EPV from both dying 0.000002863602

$x = (1+i)^{\frac{1}{12}}$

$$y = \frac{23}{144}x^{11} + \frac{21}{144}x^{10} + \frac{19}{144}x^{9} + \frac{17}{144}x^{8} + \frac{15}{144}x^{7} + \frac{13}{144}x^{6} + \frac{11}{144}x^{5} + \frac{9}{144}x^{4} + \frac{7}{144}x^{3} + \frac{5}{144}x^{2} + \frac{3}{144}x + \frac{1}{144}$$

$$y - xy = \frac{1}{144} + \frac{2}{144}(x + x^2 + \cdots + x^{11}) - \frac{23}{144}x^{12}$$

$$= \frac{1}{144} + \frac{2x}{144}\left(\frac{x^{11}-1}{x-1}\right) - \frac{23}{144}x^{12}$$

$$y = \frac{1}{144(x-1)}\left(23x^{12} - 1 - 2\frac{x^{11}-1}{x-1}\right)$$

$$= \frac{1}{12i^{(12)}}\left(23i + 22 - 2\frac{x^{11}-1}{x-1}\right)$$

$i^{(12)} = 0.039288$
$y = 1.024815$

$$A^{(12\ 1)}{}_{45,76:\overline{7}|} = 0.0138451\frac{i}{i^{(12)}} + 0.000002863602 \times 1.024815 = 0.01410006$$

$$A^{(12)}_{45,76:\overline{7}|} = 0.01410006 + 0.997090283582(1.04)^{-7} = 0.7718067$$

$$d^{(12)} = 12\left(1 - (1.04)^{-\frac{1}{12}}\right) = 0.03915669$$

This gives $\ddot{a}^{(12)}_{45,76:\overline{7}|} = \frac{1-0.7718067}{0.03915669} = 5.827696$
So the monthly premiums are $\frac{850000\times0.01410006}{12\times5.827696} = \$171.38$

9.6 A Model with Dependent Future Lifetimes

23

(a)

Year	P(Husband Dies)	P(Wife Survives)	P(Wife Survives to 49)	P(Husband Dies and Wife Survives to 49)
1	0.004599	0.999900	$\frac{9961.92}{9997.47}$	0.004582188
2	0.004984	0.999794	$\frac{9961.92}{9994.72}$	0.004966621
3	0.005398	0.999681	$\frac{9961.92}{9991.74}$	0.005380173
4	0.005844	0.999560	$\frac{9961.92}{9988.49}$	0.00582589
5	0.006324	0.999432	$\frac{9961.92}{9984.95}$	0.00630583
6	0.006840	0.999294	$\frac{9961.92}{9981.10}$	0.006822036
7	0.007394	0.999148	$\frac{9961.92}{9976.91}$	0.007376601
8	0.007989	0.998991	$\frac{9961.92}{9972.34}$	0.0079726
9	0.008625	0.998823	$\frac{9961.92}{9967.36}$	0.008610147
$\geqslant 10$	0.942003	0.998644	1	0.9407256
				0.9985677

(b)

Suppose the probability of the wife surviving while the husband is alive is p_1 and while the husband is dead is p_2. If the husband dies at time t during the year, the wife's probability of surviving the year is $(1 - tq_1)\left(\frac{p_2}{1-tq_2}\right)$

The probability of the wife's surviving the year if the husband dies at a uniformly distributed time during the year is therefore

$$\int_0^1 (1 - tq_1)\left(\frac{p_2}{1 - tq_2}\right) dt = p_2\left(\frac{q_1}{q_2} + (q_2 - q_1)\log(1 - q_2)\right)$$

Year	P(Husband Dies)	P(Wife Survives to start of year)	P(Wife survives year given Husband dies)	P(Wife Survives to 49)	P(Husband Dies and Wife Survives to 49)
1	0.004599	1	0.9998235	$\frac{9961.92}{9997.47}$	0.004581838
2	0.004984	0.999900	0.9995565	$\frac{9961.92}{9994.72}$	0.004964944
3	0.005398	0.999794	0.9992660	$\frac{9961.92}{9991.74}$	0.005376837
4	0.005844	0.999681	0.9989505	$\frac{9961.92}{9988.49}$	0.005820486
5	0.006324	0.999560	0.9986070	$\frac{9961.92}{9984.95}$	0.006297865
6	0.006840	0.999432	0.9982330	$\frac{9961.92}{9981.10}$	0.006810936
7	0.007394	0.999294	0.9978261	$\frac{9961.92}{9976.91}$	0.007361662
8	0.007989	0.999148	0.9973831	$\frac{9961.92}{9972.34}$	0.007953002
9	0.008625	0.998991	0.9969001	$\frac{9961.92}{9967.36}$	0.008584917
10	0.009305	0.998823	0.9963737	1	0.009260354
$\geqslant 10$	0.932698	0.998644	1	1	0.9407256
					0.9984461

24

By summing over times the husband dies, we calculate the following lifetable for the wife.

Suppose the mortality for the wife while the husband is alive is q_1, the mortality while the husband is dead is q_2, and the mortality of the husband is q_3. Conditional on the husband dying at time t in the year, the probability that the wife dies during the year is

$$tq_1 + (1 - tq_1)\left(1 - \frac{1 - q_2}{1 - tq_2}\right) = tq_1 + (1 - tq_1)\frac{(1 - t)q_2}{1 - tq_2}$$

The total probability that the wife dies during the year if the husband is alive at the start of the year is therefore

$$
\begin{aligned}
&(1-q_3)q_1 + \int_0^1 q_3 \left(tq_1 + (1-tq_1)\frac{(1-t)q_2}{1-tq_2} \right) dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3 \int_0^1 (1-t)\frac{1-tq_1}{1-tq_2}\, dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3 \int_0^1 \frac{1-(1+q_1)t+q_1t^2}{1-tq_2}\, dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3 \int_0^1 \frac{1+q_1+\frac{q_1}{q_2}}{q_2} - \frac{q_1}{q_2}t + \frac{1-\frac{1+q_1+\frac{q_1}{q_2}}{q_2}}{1-tq_2}\, dt \\
&= (1-q_3)q_1 + \frac{q_1q_3}{2} + q_2q_3 \left(\frac{1+q_1+\frac{q_1}{q_2}}{q_2} - \frac{q_1}{2q_2} + \left(1-\frac{1+q_1+\frac{q_1}{q_2}}{q_2}\right) \left[-\frac{\log(1-tq_2)}{q_2} \right]_0^1 \right) \\
&= (1-q_3)q_1 + q_3\left(1+q_1+\frac{q_1}{q_2}\right) + q_3\left(1-\frac{1+q_1+\frac{q_1}{q_2}}{q_2}\right)(\log(1)-\log(1-q_2)) \\
&= (1-q_3)q_1 + q_3\left(1+q_1+\frac{q_1}{q_2}\right) - q_3\left(\frac{{q_2}^2-q_2-q_1q_2-q_1}{{q_2}^2}\right)\log(1-q_2) \\
&= q_1 + q_3\left(\frac{q_1+q_2}{q_2}\right) + q_3\left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)
\end{aligned}
$$

If the probability that the husband and wife are both alive at the start of the year is p_1 and the probability that the wife is alive, but the husband is dead is p_2, then the probability that both are alive at the end of the year is $p_1(1-q_1)(1-q_3)$. The probability that the wife is alive and the husband is dead at the end of the year is therefore:

$$
\begin{aligned}
&p_2(1-q_2) + p_1\left(1 - \left(q_1 + q_3\left(\frac{q_1+q_2}{q_2}\right) + q_3\left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)\right) - (1-q_1)(1-q_3)\right) \\
&= p_2(1-q_2) + p_1\left(q_3(1-q_1) - \left(q_3\left(\frac{q_1+q_2}{q_2}\right) + q_3\left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)\right)\right) \\
&= p_2(1-q_2) - p_1q_3\left(\left(\left(q_1+\frac{q_1}{q_2}\right) + \left(\frac{q_2+q_1q_2+q_1-{q_2}^2}{{q_2}^2}\right)\log(1-q_2)\right)\right)
\end{aligned}
$$

Year	Both Alive	Husband Dead, Wife Alive	Wife Alive
0	10000.00	0.00	10000.00
1	9953.01	45.99	9999.00
2	9902.13	95.80	9997.93
3	9847.05	149.74	9996.79
4	9787.45	208.11	9995.56
5	9722.99	271.24	9994.23
6	9653.31	339.49	9992.80
7	9578.01	413.22	9991.23
8	9496.70	492.84	9989.54
9	9408.95	578.74	9987.70
10	9314.33	671.35	9985.69

This gives

$$a_{39:\overline{10}|} = 8.431041$$
$$A_{39:\overline{10}|} = 0.001132052$$

So the annual premium is

$$\frac{200000 \times 0.001132052}{8.431041} = \$26.85$$

9.7 The Common Shock Model

25

We have

$$ {}_tp_{25:56}^{(00)} = e^{-\int_0^t 0.000042+0.000000001(56+s)+0.000000002(25+s)+0.000001(25+s)^2+0.000002(56+s)^2 ds}$$
$$=e^{-\int_0^t 0.003403106+0.000174003s+0.000003s^2 ds}$$
$$=e^{-(0.003403106t+0.0000870015t^2+0.000001t^3)}$$

So the probability is given by

$$\int_0^{10} e^{-(0.003403106t+0.0000870015t^2+0.000001t^3)}(0.000002(25+t)^2+0.000000002(56+t))e^{-0.000002\frac{(66^3-(56+t)^3)}{3}}\,dt$$
$$=\int_0^{10} e^{-(0.07458667-0.002868894t-0.0000249985t^2-0.000001t^3)}(0.001250112+0.000100002t+0.000002t^2)dt$$
$$=0.01715084$$

26

If the husband dies first after time t, then the present value (at time of husband's death) of the life annuity is

$$25000\int_0^\infty e^{-0.005s}e^{-0.04s}\,ds = \frac{25000}{0.045} = 555555.56$$

The total transition intensity out of state 0 is

$$\begin{aligned}&(0.000001(56+t)^2+0.000000001(25+t))+(0.000002(25+t)^2+0.000000002(56+t))+0.000042\\&=0.000003t^2+0.000212003t+0.004428137\end{aligned}$$

The expected value of the life annuity is therefore

$$\begin{aligned}&555555.56\int_0^\infty (0.000002(25+t)^2+0.000000002(56+t))e^{-(0.000003t^2+0.000212003t+0.004428137)}e^{-0.04t}\,dt\\&=555555.56\int_0^\infty (0.000002(25+t)^2+0.000000002(56+t))e^{-(0.000003t^2+0.040212003t+0.004428137)}\,dt\\&=\$84,251.58\end{aligned}$$

(Numerically integrated)
The expected present value of the premiums is

$$\begin{aligned}&P\int_0^\infty e^{-(0.000003t^2+0.000212003t+0.004428137)}e^{-0.04t}\,dt\\&=P\int_0^\infty e^{-(0.000003t^2+0.040212003t+0.004428137)}\,dt\\&=P\int_0^\infty e^{-(0.000003(t+6702.0005)^2-134.746)}\,dt\\&=Pe^{134.746}\int_0^\infty e^{-0.000003(t+6702.0005)^2}\,dt\\&=Pe^{134.746}\sqrt{\frac{\pi}{0.000003}}\left(1-\Phi(6702.0005\sqrt{0.000006})\right)\\&=24.66746P\end{aligned}$$

We therefore get that

$$\begin{aligned}24.66746P&=84251.58\\P&=\$3,415.49\end{aligned}$$

27
We have

$$_tp_{xy}^{(00)}=e^{-\int_0^t 0.001002(y+s)+0.002001(x+s)+0.012\,ds}=e^{-((0.001002y+0.002001x+0.012)s+0.0015015s^2)}$$

Numerically integrating gives the following lifetable

Year	Both alive	H alive W dead	H dead W alive	Both dead
0	10000.00	0.00	0.00	0.00
1	8247.83	239.88	1332.93	179.36
2	6782.28	394.62	2365.22	457.88
3	5560.41	485.42	3144.19	809.98
4	4545.01	529.17	3711.21	1214.61
5	3703.89	539.19	4102.32	1654.61
6	3009.38	525.84	4348.71	2116.08
7	2437.76	497.08	4477.26	2587.90
8	1968.80	458.92	4511.01	3061.27
9	1585.29	415.81	4469.58	3529.31
10	1272.65	370.99	4369.61	3986.74
11	1018.61	326.71	4225.09	4429.59
12	812.84	284.47	4047.71	4854.98
13	646.68	245.24	3847.19	5260.88
14	512.95	209.54	3631.53	5645.98
15	405.66	177.59	3407.22	6009.54
16	319.84	149.39	3179.52	6351.25
17	251.43	124.80	2952.61	6671.16
18	197.05	103.58	2729.76	6969.60
19	153.97	85.45	2513.48	7247.10
20	119.95	70.09	2305.62	7504.34
21	93.17	57.17	2107.54	7742.12
22	72.15	46.39	1920.14	7961.32
23	55.70	37.45	1743.97	8162.88
24	42.87	30.09	1579.29	8347.75
25	32.90	24.06	1426.12	8516.91
26	25.18	19.15	1284.33	8671.34
27	19.20	15.17	1153.63	8812.00
28	14.61	11.97	1033.61	8939.81
29	11.08	9.40	923.82	9055.71
30	8.37	7.35	823.71	9160.56
31	6.31	5.73	732.75	9255.21
32	4.74	4.44	650.34	9340.48
33	3.55	3.43	575.90	9417.11
34	2.65	2.64	508.86	9485.85
35	1.98	2.03	448.64	9547.36
36	1.47	1.55	394.69	9602.29
37	1.09	1.18	346.50	9651.24
38	0.80	0.89	303.54	9694.76
39	0.59	0.67	265.36	9733.38
40	0.43	0.51	231.49	9767.57
41	0.32	0.38	201.53	9797.77
42	0.23	0.28	175.08	9824.40
43	0.17	0.21	151.80	9847.82
44	0.12	0.16	131.34	9868.38

Year	Both alive	H alive W dead	H dead W alive	Both dead
45	0.09	0.12	113.41	9886.39
46	0.06	0.09	97.73	9902.13
47	0.05	0.06	84.04	9915.85
48	0.03	0.05	72.13	9927.79
49	0.02	0.03	61.78	9938.17
50	0.02	0.02	52.81	9947.15
51	0.01	0.02	45.05	9954.92
52	0.01	0.01	38.35	9961.63
53	0.01	0.01	32.58	9967.40
54	0.00	0.01	27.63	9972.36
55	0.00	0.00	23.38	9976.61
56	0.00	0.00	19.74	9980.25
57	0.00	0.00	16.64	9983.35
58	0.00	0.00	14.00	9986.00
59	0.00	0.00	11.75	9988.25
60	0.00	0.00	9.84	9990.15
61	0.00	0.00	8.23	9991.77
62	0.00	0.00	6.87	9993.13
63	0.00	0.00	5.72	9994.28
64	0.00	0.00	4.75	9995.25
65	0.00	0.00	3.94	9996.06
66	0.00	0.00	3.26	9996.74
67	0.00	0.00	2.70	9997.30
68	0.00	0.00	2.22	9997.78
69	0.00	0.00	1.83	9998.17
70	0.00	0.00	1.50	9998.50
71	0.00	0.00	1.23	9998.77
72	0.00	0.00	1.01	9998.99
73	0.00	0.00	0.82	9999.18
74	0.00	0.00	0.67	9999.33
75	0.00	0.00	0.54	9999.46
76	0.00	0.00	0.44	9999.56
77	0.00	0.00	0.36	9999.64
78	0.00	0.00	0.29	9999.71
79	0.00	0.00	0.23	9999.77
80	0.00	0.00	0.19	9999.81
81	0.00	0.00	0.15	9999.85
82	0.00	0.00	0.12	9999.88
83	0.00	0.00	0.10	9999.90
84	0.00	0.00	0.08	9999.92
85	0.00	0.00	0.06	9999.94
86	0.00	0.00	0.05	9999.95
87	0.00	0.00	0.04	9999.96
88	0.00	0.00	0.03	9999.97
89	0.00	0.00	0.02	9999.98

(a) Summing up we get that $A_{\overline{75,29}} = 0.4811416$. This gives $a_{\overline{75,29}} = \frac{1-0.4811416}{0.06} = 8.64764$. so the premium is $300000 \times 0.48114168.64764 = \$16,691.55$.

(b) After 10 years, if the husband is dead, but the wife is alive, then the mortality is $\mu_y^{23} = 0.002y$. The probability that she survives for s years is therefore

$$e^{-\int_0^s 0.002(39+s)\,ds} = e^{-0.078s-0.001s^2}$$

This gives $A_{39} = \sum_{i=1}^{\infty}(1.06)^{-i}(e^{-0.078(i-1)-0.001(i-1)^2} - e^{-0.078i-0.001i^2}) = 0.590609$, so $\ddot{a}_{39} = \frac{1-0.590609}{0.06} = 6.823183$. This means that the policy value is

$$300000 \times 0.590609 - 16691.55 \times 6.823183 = \$63,293.20$$

SN 4.2 Mortality Improvement Scales

28

We first interpolate the age effect. That is, we set $\phi(45, 2000+t) = f_{45}(t) = at^3 + bt^2 + ct + d$. We have

$$\begin{aligned} f(0) &= 0.01863027 \\ f'(0) &= 0.001077732 \\ f(25) &= 0.01 \\ f'(25) &= 0 \end{aligned}$$

Substituting the above formula gives the equations

$$\begin{aligned} d &= 0.01863027 && (1) \\ c &= 0.001077732 && (2) \\ 15625a + 625b + 25c + d &= 0.01 && (3) \\ 1875a + 50b + c &= 0 && (4) \end{aligned}$$

subtracting twice (3) from 25 times (4) gives

$$\begin{aligned} 15625a &= 25c + d \\ &= 0.04557357 \\ a &= 0.000002916708 \\ b &= -0.0001309312 \end{aligned}$$

This gives $f(13) = 0.000002916708(13)^3 - 0.0001309312(13)^2 + 0.001077732(13) + 0.01863027 = 0.01692142$

For the cohort effect, we set $\phi_*(32+t, 2000+t) = g(t) = a^*t^3 + b^*t^2 + c^*t + d^*$. Setting up the same equations gives

$$\begin{aligned} d^* &= 0.03146243 && (5) \\ c^* &= 0.001847864 && (6) \\ 15625a^* + 625b^* + 25c^* + d^* &= 0.01 && (7) \\ 1875a^* + 50b^* + c^* &= 0 && (8) \end{aligned}$$

subtracting twice (7) from 25 times (8) gives

$$\begin{aligned}15625a &= 25c + d\\ &= 0.07765903\\ a &= 0.000004970178\\ b &= -0.000223339\end{aligned}$$

Substituting these gives $\phi_*(45, 2013) = g(13) = 0.000004970178(13)^3 - 0.000223339(13)^2 + 0.001847864(13) + 0.03146243 = 0.02865985$

The overall scale function is the average of these two values $\phi(45, 2013) = \frac{0.01692142+0.02865985}{2} = 0.02279064$.

The Lee Carter Model

29

(a) We have $\log(m(40, 2034)) = \alpha_{40} + \beta_{40}K_{2034}$. We also have that $K_{2034} = K_{2017} + 17c + \sigma_k \sum_{i=0}^{16} Z_{2017+i}$. A sum of normal distributions is normal, so K_{2034} follows a normal distribution with mean $K_{2017}+17c$ and variance $17{\sigma_k}^2$. Substituting this into the above formula gives us $\log(m(40, 2034)) \sim N(-5.7672, 0.029988)$. The mean of a log-normal distribution with parameters μ and σ^2 is $e^{\mu+\frac{\sigma^2}{2}} = e^{-5.7672+\frac{0.029988}{2}} = 0.003175767$, while the expected square of the log-normal distribution is $e^{2\mu+2\sigma^2} = e^{-2\times5.7672+2\times0.029988} = 0.00001039252$. The variance of $m(40, 2034)$ is therefore $0.00001039252 - 0.003175767^2 = 3.07024 \times 10^{-07}$, so the standard deviation is $\sqrt{3.07024 \times 10^{-07}} = 0.0005540974$. The 5th percentile of $\log(m(40, 2034))$ is $-5.7672 + \Phi^{-1}(0.05)\sqrt{0.029988} = -6.05204$. Therefore the 5th percentile of $m(40, 2034)$ is $e^{-6.05204} = 0.002353057$.

(b) The life will be 40 in 7 years, so we need to calculate $m(40, 2024)$. Using the same methods as in part (a), we get that $\log(m(40, 2024)) \sim N(-5.2072, 0.012348)$.

Under the UDD assumption, suppose we are given q_x, we have ${}_tp_x = 1 - tq_x$, so $\int_0^1 {}_tp_x\,dt = \int_0^1 (1 - tq_x)\,dt = 1 - \frac{q_x}{2}$. This gives

$$\begin{aligned}m_x &= \frac{q_x}{1-\frac{q_x}{2}}\\ &= \frac{2q_x}{2-q_x}\\ \frac{1}{m_x} &= \frac{1}{q_x} - \frac{1}{2}\\ \frac{1}{q_x} &= \frac{1}{m_x} + \frac{1}{2}\\ &= \frac{2+m_x}{2m_x}\\ q_x &= \frac{2m_x}{2+m_x}\end{aligned}$$

We have

$$\log\left(\frac{\frac{q_x}{2}}{1-\frac{q_x}{2}}\right) = \log\left(\frac{m_x}{2}\right) \sim N(-5.2072 - \log(2), 0.012348)$$

so $\frac{q_x}{2}$ has a logit-normal distribution. There is no analytic solution for the mean, but numerically, we can get $\mathbb{E}(q_x) = 0.00550$. Using this, the premium is $300000(1.05)^{-1} \times 0.0054955 = \$1,570.16$.

(c) The EPVFL of the policy is $300000q(40,2024)(1.05)^{-1} - 1570.16$, so the EPVFL exceeds \$50 if

$$\begin{aligned}
300000q(40,2024)(1.05)^{-1} - 1570.16 &\geqslant 50 \\
300000q(40,2024)(1.05)^{-1} &\geqslant 1620.16 \\
q(40,2024) &\geqslant \frac{1620.16}{300000(1.05)^{-1}} \\
m(40,2024) &\geqslant \frac{1620.16 \times 2.1}{2 \times 300000 - 1620.16 \times 1.05} \\
&= 0.005686683 \\
\log(m(40,2024)) &\geqslant -5.169628 \\
\frac{\log(m(40,2024)) + 5.2072}{0.012348} &\geqslant 3.042752
\end{aligned}$$

so the probability is $1 - \Phi^{-1}(3.042752) = 0.001172127$.

Cairns-Blake-Dowd Models

30

(a) We have that

$$\begin{aligned}
\log\left(\frac{q(33,2048)}{1-q(33,2048)}\right) &= K_{2048}^{(1)} - 19K_{2048}^{(2)} \\
&= K_{2017}^{(1)} - 19K_{2017}^{(2)} + 31(c^{(1)} - 19c^{(2)}) + \sum_{i=1}^{31}(\sigma_{k_1}Z_{2017+i}^{(1)} - 19\sigma_{k_2}Z_{2017+i}^{(2)}) \\
&= -7.14 + \sum_{i=1}^{31}(\sigma_{k_1}Z_{2017+i}^{(1)} - 19\sigma_{k_2}Z_{2017+i}^{(2)})
\end{aligned}$$

Now we have that

$$\mathrm{Var}(\sigma_{k_1}Z_{2017+i}^{(1)} - 19\sigma_{k_2}Z_{2017+i}^{(2)}) = {\sigma_{k_1}}^2 + 361{\sigma_{k_2}}^2 - 38\sigma_{k_1}\sigma_{k_2}\rho = 0.3481$$

so

$$\mathrm{Var}(\sum_{i=1}^{31}(\sigma_{k_1}Z_{2017+i}^{(1)} - 19\sigma_{k_2}Z_{2017+i}^{(2)})) = 31 \times 0.3481 = 10.7911$$

so

$$\log\left(\frac{q(33,2048)}{1-q(33,2048)}\right) \sim N(-7.14, 10.7911)$$

The median is therefore given by solving

$$
\begin{aligned}
\log\left(\frac{q(33,2048)}{1-q(33,2048)}\right) &= -7.14 \\
\frac{q(33,2048)}{1-q(33,2048)} &= 0.0007927521 \\
\frac{1}{q(33,2048)} - 1 &= 1261.428 \\
\frac{1}{q(33,2048)} &= 1262.428 \\
q(33,2048) &= 0.0007921241
\end{aligned}
$$

The 95th percentile of $\log\left(\frac{q(33,2048)}{1-q(33,2048)}\right)$ is $-7.14 + \Phi^{-1}(0.95)\sqrt{10.7911} = -1.736687$, so the 95th percentile of $q(33,2048)$ is given by solving

$$
\begin{aligned}
\log\left(\frac{q(33,2048)}{1-q(33,2048)}\right) &= -1.736687 \\
\frac{q(33,2048)}{1-q(33,2048)} &= 0.1761028 \\
\frac{1}{q(33,2048)} - 1 &= 5.6785 \\
\frac{1}{q(33,2048)} &= 6.6785 \\
q(33,2048) &= 0.1497342
\end{aligned}
$$

(b) From the simulated values we get:

t	201t	2018	2019	2020	2021
$K_t^{(1)}$	-4.36	-5.203697	-5.323657	-6.351717	-7.493481
$K_t^{(2)}$	0.13	0.08674009	0.02933806	-0.02750201	-0.06004049

This gives us

$$\begin{aligned}
\log\left(\frac{q(26,2017)}{1-q(26,2017)}\right) &= -4.36 - 0.13\times(52-26) = -7.740000\\
\log\left(\frac{q(27,2018)}{1-q(27,2018)}\right) &= -5.203697 - 0.08674009\times(52-27) = -7.372200\\
\log\left(\frac{q(28,2019)}{1-q(28,2019)}\right) &= -5.323657 - 0.02933806\times(52-28) = -6.027771\\
\log\left(\frac{q(29,2020)}{1-q(29,2020)}\right) &= -6.351717 + 0.02750201\times(52-29) = -5.719170\\
\log\left(\frac{q(30,2021)}{1-q(30,2021)}\right) &= -7.493481 + 0.06004049\times(52-30) = -6.172590
\end{aligned}$$

and thus

$$\begin{aligned}
q(26,2017) &= 0.0004348824\\
q(27,2018) &= 0.0006280895\\
q(28,2019) &= 0.0024050639\\
q(29,2020) &= 0.0032716939\\
q(30,2021) &= 0.0020814853
\end{aligned}$$

For these values we get $\ddot{a}_{26:\overline{5}|} = 4.536046$ and $A_{26:\overline{5}|} = 1 - \frac{0.05}{1.05}\times 4.536046 = 0.7839978$, and ${}_5p_{26} = 0.9912070$, so $A_{26:\overline{5}|} = 0.7839978 - 0.9912070(1.05)^{-5} = 0.007361232$. The premium is therefore

$$\frac{400000\times 0.007361232}{4.536046} = \$649.13$$

LM 12 Empirical Estimation

LM 12.1 The Empirical Distribution

31

The probability mass function is

n	$P(X=n)$
0	0.2
1	0.1333
2	0.0667
3	0.2667
4	0.1333
6	0.1333
7	0.0667

The cumulative Hazard rate is

x	$H(x)$	$H(x)$
0	0.2231436	0.2
1	0.4054651	0.3667
2	0.5108256	0.4667
3	1.0986123	0.9111
4	1.6094379	1.3111
6	2.7080502	1.9778
7		2.9778

32

The Nelson-Åalen estimate is $H(5) = 1.31111111$, so this gives $S(5) = e^{-1.311111} = 0.2695204$.

LM 12.2 The Empirical Distributions for Grouped Data

33

The total number of policies is 1099. 194 are less than \$100,000, and 558 are less than \$500,000, so the empirical estimates are $F(100000) = \frac{194}{1099}$ and $F(500000) = \frac{558}{1099}$. The ogive then gives $F(300000) = \frac{1}{2}\left(\frac{194}{1099} + \frac{558}{1099}\right) = \frac{376}{1099} = 0.3421292$. So the probability that a random policy would be affected by this tax is 0.6578708.

34

See slides.

35

over the 2000 observations, the total of all values of $X \wedge 6000$ is $2000 \times 1810 = 3,620,000$. There are 300 observations for which $X \wedge 6000 = 6000$. The sum of these is therefore $300 \times 6000 = 1,800,000$. The total of the 1700 observations where X is less than $6,000$ is therefore $3,620,000 - 1,800,000 = 1,820,000$. The total of the 30 observations between 6,000 and 7,000 is 200,000, so the total of the 1,730 observations below 7,000 is $2,020,000$. The total of $X \wedge 7000$ for the 270 observations above 7,000 is $7000 \times 270 = 1,890,000$ so the total of all 2000 observations of $X \wedge 7000$ is $2020000 + 1890000 = 3910000$, so the average $\mathbb{E}(X \wedge 7000) = \frac{3910000}{2000} = 1,955$.

36

The total number of observations is $200 + x + y$. The number of observations less than 50 is 36. The number of observations less than 150 is $36 + x$. The number of observations less than 250 is $36 + x + y$. Therefore

$$\begin{aligned}
F_n(50) &= \frac{36}{200 + x + y} \\
F_n(150) &= \frac{36 + x}{200 + x + y} \\
F_n(250) &= \frac{36 + x + y}{200 + x + y} \\
F_n(90) &= \frac{36 + 0.4x}{200 + x + y} = 0.21 \\
F_n(210) &= \frac{36 + x + 0.6y}{200 + x + y} = 0.51
\end{aligned}$$

We therefore need to solve the equations

$$
\begin{aligned}
36+0.4x &= 0.21(200+x+y)\\
19x-21y &= 600\\
36+x+0.6y &= 0.51(200+x+y)\\
49x+9y &= 6600\\
1200x &= 144000\\
x &= 120\\
y &= 80
\end{aligned}
$$

37

Suppose we are estimating the survival function at x which is in the interval $(c_1, c_2]$. The estimate is

$$S(x) = \frac{c_2 - x}{c_2 - c_1} S(c_1) + \frac{x - c_1}{c_2 - c_1} S(c_2)$$

Let X be the number of observations from a sample of n observations that are less than c_1, and let Y be the number that are between c_1 and c_2. We then have

$$\hat{S}(x) = \frac{c_2 - x}{c_2 - c_1}\frac{n - X}{n} + \frac{x - c_1}{c_2 - c_1}\frac{n - X - Y}{n} = 1 - \frac{X(c_2 - x) + (X+Y)(x - c_1)}{c_2 - c_1} = 1 - \frac{X}{n} - \frac{x - c_1}{c_2 - c_1}Y$$

We therefore have that

$$\mathrm{Var}(\hat{S}(x)) = \frac{\mathrm{Var}(X) + \left(\frac{x-c_1}{c_2-c_1}\right)^2 \mathrm{Var}(Y) + 2\left(\frac{x-c_1}{c_2-c_1}\right)\mathrm{Cov}(X,Y)}{n^2}$$

We also have that X and Y are multinomially distributed with probabilities $1 - S(c_1)$ and $S(c_1) - S(c_2)$ respectively. This means

$$
\begin{aligned}
\mathrm{Var}(X) &= nS(c_1)(1 - S(c_1))\\
\mathrm{Var}(Y) &= n(S(c_1) - S(c_2))(1 + S(c_2) - S(c_1))\\
\mathrm{Cov}(X,Y) &= -n(1 - S(c_1))(S(c_1) - S(c_2))
\end{aligned}
$$

This gives that

$$\mathrm{Var}(\hat{S}(x)) = \frac{(c_2 - c_1)^2 S(c_1)(1 - S(c_1)) - 2(c_2 - c_1)(x - c_1)(1 - S(c_1))(S(c_1) - S(c_2)) + (x - c_1)^2(S(c_1) - S(c_2))(1 + S(c_2) - S(c_1}{n(c_2 - c_1)^2}$$

$(c_2-c_1)(1-S(c_1))\big((c_2-c_1)S(c_1)-(x-c_1)(S(c_1)-S(c_2)\big)+(x-c_1)(S(c_1)-S(c_2))\big((x-c_1)(1+S(c_2)-S(c_1))-(c_2-c_1)(1-S(c_1))\big)$

$(c_2-c_1)(1-S(c_1))\big((c_2-x)S(c_1)+(x-c_1)S(c_2)\big)+(x-c_1)(S(c_1)-S(c_2))\big((x-c_1)(1-S(c_1))-(c_2-x)(1-S(c_1))\big)$

38

We compute $\hat{S}(10000) = \frac{1446}{4356}$ and $\hat{S}(100000) = \frac{683}{4356}$. This gives $\hat{S}(50000) = \frac{5}{9} \times \frac{1446}{4356} + \frac{4}{9} \times \frac{683}{4356} = \frac{9962}{9\times 4356} =$ 0.2541067

The variances are given by

$$\begin{aligned}
\mathrm{Var}(\hat{S}(10000)) &= \frac{1446 \times 2910}{4356^3} \\
\mathrm{Var}(\hat{S}(100000)) &= \frac{683 \times 3673}{4356^3} \\
\mathrm{Cov}(\hat{S}(10000), \hat{S}(100000)) &= -\frac{2910 \times 683}{4356^3} \\
\mathrm{Var}(\hat{S}(50000)) &= \frac{9^2 \times 1446 \times 2910 + 4^2 \times 683 \times 3673 - 2 \times 9 \times 4 \times 1446 \times 683}{4356^3 \times 9^2} = \frac{237873044}{4356^3 \times 81} \\
&= 0.00003553011
\end{aligned}$$

The standard deviation is $\sqrt{0.00003553011} = 0.005960714$.

A 95% confidence interval is therefore $0.2541067 \pm 1.96 \times 0.005960714 = [0.2424237, 0.2657897]$.

LM 12.3&12.5 Empirical Estimation with Modified Data

39

The probability that a randomly chosen individual survives to more than 1.6 is expressed as the product

$$\frac{11}{12} \times \frac{13}{14} \times \frac{15}{16} \times \frac{14}{15} \times \frac{11}{13} \times \frac{9}{11} \times \frac{7}{8} = \frac{11 \times 9 \times 7}{16 \times 12 \times 8} = \frac{231}{512} = 0.451171925$$

40

$$\begin{aligned}
\frac{8}{9} &= \frac{8}{9} \geqslant \frac{1}{2} \\
\frac{8}{9} \times \frac{8}{9} &= \frac{64}{81} \geqslant \frac{1}{2} \\
\frac{64}{81} \times \frac{10}{12} &= \frac{160}{243} \geqslant \frac{1}{2} \\
\frac{160}{243} \times \frac{10}{11} &= \frac{1600}{2673} \geqslant \frac{1}{2} \\
\frac{1600}{2673} \times \frac{8}{10} &= \frac{1280}{2673} < \frac{1}{2}
\end{aligned}$$

So the median is $y_5 = 0.8$.

41

The cumulative hazard rate funtion is given by $H(1.6) = \frac{1}{12} + \frac{1}{14} + \frac{1}{16} + \frac{1}{15} + \frac{1}{13} + \frac{2}{11} + \frac{1}{8} = \frac{17160+15015+16016+18480+43680+30030}{240240} = \frac{140381}{240240}$. The survival function is therefore $S(1.6) = e^{-\frac{140381}{240240}} = 0.5574756$

42

The Kaplan-Meier estimator gives $S_n(0.5) = \frac{8}{9} \times \frac{8}{9} \times \frac{10}{12} = \frac{160}{243}$ and $S_n(1) = \frac{8}{9} \times \frac{8}{9} \times \frac{10}{12} \times \frac{10}{11} \times \frac{8}{10} = \frac{1280}{2673}$

So the conditional probability is

$$\frac{\frac{160}{243}-\frac{1280}{2673}}{\frac{160}{243}}=\frac{3}{11}$$

43

Let the dying times be $t_1,\ldots,t_n$, and let the corresponding risk sets be $r_1,\ldots,r_n$. Let the number of people surviving at each dying time be X_i. Suppose that the true probability of surviving at time t_i is p_i. The Kaplan-Meier estimate of the survival probability is therefore $\prod_{i=1}^n \frac{X_i}{r_i}$. Since the X_i are independent, we have

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^n \frac{X_i}{r_i}\right) &= \prod_{i=1}^n \mathbb{E}\left(\frac{X_i}{r_i}\right)\\ &= \prod_{i=1}^n p_i\end{aligned}$$

so the Kaplan-Meier estimate is unbiassed.

We also have:

$$\begin{aligned}\mathbb{E}\left(\prod_{i=1}^n \frac{X_i}{r_i}\right)^2 &= \prod_{i=1}^n \mathbb{E}\left(\frac{X_i}{r_i}\right)^2\\ &= \prod_{i=1}^n \left(\frac{p_i(1-p_i)}{r_i}+p_i^2\right)\end{aligned}$$

so the variance is

$$\prod_{i=1}^n \left(\frac{p_i(1-p_i)}{r_i}+p_i^2\right)-\prod_{i=1}^n {p_i}^2=\left(\prod_{i=1}^n p_i\right)^2\left(\prod_{i=1}^n\left(1+\frac{(1-p_i)}{p_ir_i}\right)-1\right)$$

If we let s_i be the total survivial probability up to time i, so that $s_i=\prod_{j=1}^i p_j$, then this becomes

$${s_n}^2\left(\prod_{i=1}^n\left(1+\frac{(s_{i-1}-s_i)}{s_ir_i}\right)-1\right)$$

44

Greenwoods formula gives that the variance is

$$\begin{aligned}
\mathrm{Var}(S_n(y_j)) &\approx \hat{S}(y_j)^2 \sum_{i=1}^{j} \frac{s_i}{r_i(r_i - s_i)} \\
&= \left(\frac{1280}{2673}\right)^2 \left(\frac{1}{9 \times 8} + \frac{1}{9 \times 8} + \frac{2}{12 \times 10} + \frac{1}{11 \times 10} + \frac{2}{10 \times 8}\right) \\
&= \left(\frac{1280}{2673}\right)^2 \left(\frac{55 + 55 + 66 + 36 + 99}{3960}\right) \\
&= 0.0180089
\end{aligned}$$

The 95% confidence interval is therefore

$$0.4788627 \pm 1.96\sqrt{0.0180089} = [0.2158361, 0.7418893]$$

45
The confidence interval is

$$[S_n(1)^{\frac{1}{U}}, S_n(1)^U]$$

where

$$U = e^{1.96 \frac{\sqrt{0.0180089}}{0.4788627 \log(0.4788627)}} = 0.4742837$$

So the confidence interval is

$$[0.4788627^{2.108443}, 0.4788627^{0.4742837}] = [0.2117109, 0.7052276]$$

46
The Nelson-Åalen estimator is $H(5) = \frac{226}{1641} + \frac{387}{1415} + \frac{290}{1028} + \frac{215}{738} + \frac{176}{523} = 1.321168$
The variance of this estimator is then $\frac{226}{1641^2} + \frac{387}{1415^2} + \frac{290}{1028^2} + \frac{215}{738^2} + \frac{176}{523^2} = 0.001589823$
We therefore have

$$\begin{aligned}
\log(\hat{H}(5)) &= 0.2785162 \\
\mathrm{Var}(\log(\hat{H}(5))) &= \frac{0.001589823}{1.321168^2} = 0.0009108203
\end{aligned}$$

So a 95% confidence interval for $\log(H(5))$ is

$$0.2785162 \pm 1.96\sqrt{0.0009108203} = [0.2193638, 0.3376686]$$

The corresponding interval for $H(5)$ is

$$[1.245284, 1.401676]$$

and the corresponding interval for $S(5)$ is

$$[0.2461840, 0.2878591]$$

LM 12.7 Approximations for Large Data Sets

47

(a) The exact exposure is $1 + 0.7 + 1 + 0.2 + 0.8 + 1 + 0.4 + 1 + 0.4 + 0.8 + 0.2 + 1 + 0.4 + 1 + 0.5 + 0.1 + 0.9 + 0.6 + 0.2 + 0.4 = 12.2$ years. There are two deaths in the interval. The estimate for the hazard rate is therefore $\frac{3}{12.2} = 0.2459016$, and the probability of dying in the year is $1 - e^{-0.2459016} = 0.2180008$.

(b) The actuarial exposure is $1+0.7+1+0.2+0.8+1+0.4+1+0.4+0.8+0.2+1+0.4+1+0.5+0.1+0.9+0.6+1+1 = 13.6$, so the estimate for the probability of dying is $\frac{3}{13.6} = 0.2205882$.

48

Using insuring ages, the table looks like this:

entry	death	exit	entry	death	exit
61.2	-	64.2	63.0	-	64.0
61.7	-	63.0	61.8	-	64.0
62.4	-	64.1	61.4	-	63.0
60.1	-	62.3	62.6	-	65.6
62.8	-	65.8	61.0	62.4	-
62.0	-	64.3	62.0	63.2	-
63.6	-	66.6	62.0	64.9	-
61.7	-	64.7	62.1	-	63.5
60.2	-	63.0	62.2	62.7	-
60.4	-	62.9	62.8	65.0	-

(a) Now the exact exposure is given by $1+0+1+0+1+1+0.4+1+0+0+1+1+0+1+0+0.2+1+0.5+0+1 = 11.1$, so the estimated hazard rate is $\frac{1}{11.1} = 0.09009009$ and the estimated probability of dying is $1 - e^{-0.09009009} = 0.08615115$.

The actuarial exposure is given by $1+0+1+0+1+1+0.4+1+0+0+1+1+0+1+0+1+1+0.5+0+1 = 11.9$ so the estimated probability of dying is $\frac{1}{11.9} = 0.08403361$.

(b) Using an anniversary-to-anniversary study, we ignore all partial units of exposure, so the exposure is 11, which makes $q_{63} = \frac{1}{11} = 0.0909$.

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See next slide.

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(a) The exact exposure is $\frac{15+11}{2} = 13$. The hazard rate is therefore $\frac{3}{13} = 0.2307692$ and the probability of dying during the year is therefore $1 - e^{-0.2307692} = 0.2060773$.

(b) The actuarial exposure is $15 + \frac{5-6}{2} = 14.5$ and the probability of dying during the year is therefore $\frac{3}{14.5} = 0.2068966$.

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Now death is the censoring event and withdrawl is the event we are trying to estimate. The exposure is $15 + \frac{5-3}{2} = 16$ and the probability of withdrawing is therefore $\frac{6}{16} = 0.375$.

LM 12.9 Estimation of Transition Intensities

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MLE estimates are based on exact exposure. We calculate the total exposure in each state for the data:

State	Exposure	Transition to	No. observed	Intensity
Healthy	$1+1+1+0.6+0.8+0.4+0.3+0.4+0.2+0.8+0.6+0.2+0.6+(0.3+0.2)+0.5+0.1+0+0+0.4+0=9.4$	Disabled	3	0.3191489
		Surrender	3	0.3191489
		Dead	4	0.4255319
Disabled	$0+0+0+0+0+0.6+0+0+0+0+0.3+0+0+0.2+0+0+0.6+1+0.6+0.5=3.8$	Healthy	2	0.5263158
		Surrender	0	0
		Dead	2	0.5263158

10.3 The Salary Scale Function

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(a) Average Salary from age 62–65 is given by $\frac{60000(1.03^{20}+1.03^{21}+1.03^{22})}{3} = \$111,650.18$

(b) Average Salary from age 62–65 is given by $\frac{60000(2.261+2.335+2.400)}{3} = \$139,920$

(c) for scale in (a) we have $\frac{60000(1.03^{19.6666667}+1.03^{20.6666667}+1.03^{21.6666667})}{3} = \$110,445.50$

for scale in (b), we can use linear interpolation to estimate $s_{42.33333333} = \frac{2}{3}\times 1+\frac{1}{3}\times 1.036 = 1.012$. This gives a final average salary of $\frac{60000(2.261+2.335+2.400)}{3\times 1.012} = \$138,260.87$

10.4 Setting the DC Contribution

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If current salary is 1, final average salary is $\frac{1.03^{32}+1.03^{33}+1.03^{34}}{3} = 2.653108$. The replacement ratio means the original annuity is worth 1.591865, and the reversionary annuity is worth 0.7959323.

$$_tp_{65} = e^{-\int_0^t 0.000002e^{\log(1.093)(65+t)}dt} = e^{-0.0006476502\left[\frac{e^{\log(1.093)s}}{\log(1.093)}\right]_0^t} = e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}$$

The value of the life annuity is given by

$$\overline{a}_{65} = \int_0^\infty e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 21.07607$$

The value of the reversionary annuity is given by

$$\overline{a}_{65|62} = \int_0^\infty \left(1-e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 1.416998$$

So the EPV of the benefits at the time of retirement is

$$1.591865\times 21.07607+0.7959323\times 1.416998 = 34.67809$$

If first monthly salary is x, then we have

$$x(1+1.03^{\frac{1}{12}}+\ldots+1.03^{\frac{11}{12}}) = 1$$
$$x\frac{1.03-1}{1.03^{\frac{1}{12}}-1} = 1$$
$$x = \frac{1.03^{\frac{1}{12}}-1}{0.03} = 0.08220899$$

The accumulated value of all salary paid monthly in arrear at the end of 35 years is

$$0.08220899\sum_{i=1}^{420}(1.03)^{\frac{i}{12}}(1.06)^{35-\frac{i}{12}} = 0.08220899(1.03)^{35}\sum_{i=1}^{420}\left(\frac{1.06}{1.03}\right)^{\frac{i}{12}} = 0.08220899(1.03)^{35}\frac{\left(\frac{1.06}{1.03}\right)^{35}-1}{\left(\frac{1.06}{1.03}\right)^{\frac{1}{12}}-1} = 167.2143$$

So the percentage of salary needed each month is

$$\frac{34.67809}{167.2143} = 20.74\%$$

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For questions (a), (b), (e), (f) and (g), the employee's accumulated total is still 34.67809, and the final average salary is 2.653108.

For (a), the replacement ratio is $\frac{34.67809}{1.591865\times 21.07607}\times 60\% = 62.02\%$

For (b), the value of the reversionary annuity is

$$\overline{a}_{65|62} = \int_0^\infty \left(1-e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right)e^{-0.01483447\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 0.9445889$$

so the replacement ratio is $\frac{34.67809}{1.591865\times 21.07607+0.7959323\times 0.9445889}\times 60\% = 60.66\%$

For (e) the replacement ratio is $\frac{34.67809}{1.591865\times 21.07607+0.4775595\times 1.416998}\times 60\% = 60.79\%$

For (f) we have

$$\overline{a}_{65} = \int_0^\infty e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}e^{-log(1.03)t}dt = 25.05707$$

The value of the reversionary annuity is given by

$$\overline{a}_{65|62} = \int_0^\infty \left(1-e^{-0.007283007\left(e^{\log(1.093)t}-1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 2.080705$$

so the replacement ratio is $\frac{34.67809}{1.591865\times 25.05707+0.7959323\times 2.080705}\times 60\% = 50.08\%$

For (g) we have

$$\overline{a}_{65} = \int_0^\infty e^{-0.08872485\left(e^{\log(1.143)t}-1\right)}e^{-log(1.04)t}dt = 11.24322$$

and

$$\overline{a}_{65|62} = \int_0^\infty \left(1-e^{-0.08872485\left(e^{\log(1.143)t}-1\right)}\right)e^{-0.005577637\left(e^{\log(1.093)t}-1\right)}e^{-log(1.04)t}dt = 10.30211$$

so the replacement ratio is $\frac{34.67809}{1.591865\times 11.24322+0.7959323\times 10.30211}\times 60\% = 79.73\%$

For (c), the accumulated value of the investments is

$$0.08220899(1.03)^{35}\frac{\left(\frac{1.07}{1.03}\right)^{35}-1}{\frac{1.07}{1.03}-1} = 203.264$$

So the replacement ratio is $\frac{203.264}{167.2143}\times 60\% = 72.94\%$.

For (e), the accumulated value of the investments is

$$0.08220899(1.05)^{35}\frac{\left(\frac{1.06}{1.05}\right)^{35}-1}{\left(\frac{1.06}{1.05}\right)^{\frac{1}{12}}-1}=225.7627$$

but the final average salary is $\frac{1.05^{32}+1.05^{33}+1.05^{34}}{3}=5.007159$
So the new replacement ratio is $\frac{225.7627\times 2.653108}{167.2143\times 5.007159}\times 60\%=42.92\%$.

10.5 The Service Table

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(a) See slide

(b) $\frac{62.28}{4882.68}=0.01275529$

10.6 Valuation of Benefits

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(a)

$$\ddot{a}_{65}^{(12)}=\frac{1}{12}\sum_{n=0}^{\infty}(1.05)^{-\frac{n}{12}}e^{-\int_0^{\frac{n}{12}}0.0000023(1.12)^{65+t}\,dt}=\frac{1}{12}\sum_{n=0}^{\infty}(1.05)^{-\frac{n}{12}}e^{-0.0000023(1.12)^{65}\frac{(1.12)^{\frac{n}{12}}-1}{\log(1.12)}}=14.18721$$

The member's final average salary is $76000\frac{(1.04)^{16}+(1.04)^{17}+(1.04)^{18}}{3}=\$148,116.36$.
The EPV of the accrued benefit conditional on the individual retiring at age 65 is therefore:

$$148116.36\times 13\times 0.01\times 14.18721(1.05)^{-19}=\$108,105.23$$

(b)

$$\ddot{a}_{60}^{(12)}=\frac{1}{12}\sum_{n=0}^{\infty}(1.05)^{-\frac{n}{12}}e^{-0.0000023(1.12)^{60}\frac{(1.12)^{\frac{n}{12}}-1}{\log(1.12)}}=15.38250094$$

The member's final average salary is $76000\frac{(1.05)^{11}+(1.04)^{12}+(1.04)^{13}}{3}=\$121,740.85$.
The EPV of the accrued benefit conditional on the individual retiring at age 60 is therefore:

$$121740.85\times 13\times 0.01\times 15.38250094(1.05)^{-14}=\$122,957.90$$

(c)

age	probability of retirement	EPV of benefits	Probability times EPV
60	0.3	122957.90	36887.37
60.5	0.09750441650	121488.95	11845.71
61.5	0.08392282902	118539.50	9948.17
62.5	0.07223304834	115575.19	8348.35
63.5	0.06217156087	112596.85	7000.32
64.5	0.05351155835	109605.47	5865.16
65	0.3306565869	108105.23	35745.71
total			115640.78

So the EPV of accrued benefits is \$115,640.78.

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We calculate ${}_tp_{65} = e^{-\int_0^t 0.000002(1.102)^{65+t}\,dt} = e^{-0.000002(1.102)^{65}\left(\frac{1.102^t-1}{\log(1.102)}\right)} = e^{-0.01136305(1.102^t-1)}$

$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty} e^{-0.01136305(1.102^{\frac{n}{12}}-1)}(1.04^{-\frac{n}{12}} = 19.69323$

(a) If he withdraws today, he receives an annual pension of $75000\left(\frac{1.05^{-2}+1.05^{-1}+1}{3}\right)\times 0.02\times 15\times 1.02^{22} =$ $\$33,154.43$

The EPV of this is $33154.43 \times 19.69323 \times_{22} p_{43}(1.04)^{-22}$

We have ${}_{22}p_{43} = e^{-\int_0^{22} 0.000002(1.102)^{43+t}\,dt} = e^{-0.000002(1.102)^{43}\left(\frac{1.102^{22}-1}{\log(1.102)}\right)} = 0.9900282$

So the EPV is $\$272,754.94$

(b) If the employee withdraws after t years,then his annual salary is $75000(1.05)^t$, so his accrued withdrawl benefits have present value $75000(1.05)^t \times 0.02 \times 15(1.02)^{22-t} \times 19.69323_{22-t}p_{43+t}(1.04)^{-22} = \frac{272754.94}{{}_tp_{43}}$. The probability density of withdrawl after t years is $e^{-0.07(43+t)}{}_tp_{43}$

The EPV of the accrued withdrawl benefits paid upon withdrawl before age 60 is

$$\begin{aligned}
\int_0^{17} \frac{272754.94}{{}_tp_{43}}\left(\frac{1.05}{1.02}\right)^t e^{-0.07(43+t)}{}_tp_{43}\,dt &= 272754.94\int_0^{17}\left(\frac{1.05}{1.02}\right)^t e^{-0.07(43+t)}\,dt\\
&= 272754.94e^{-3.01}\int_0^{17} e^{(\log(1.05)-\log(1.02))t}e^{-0.07t}\,dt\\
&= 272754.94e^{-3.01}\int_0^{17} e^{(\log(1.05)-\log(1.02)-0.07)t}\,dt\\
&= 272754.94e^{-3.01}\left[\frac{e^{(\log(1.05)-\log(1.02)-0.07)t}}{\log(1.05)-\log(1.02)-0.07}\right]_0^{17}\\
&= 272754.94e^{-3.01}\frac{e^{17(\log(1.05)-\log(1.02)-0.07)}-1}{\log(1.05)-\log(1.02)-0.07}\\
&= \$164572.97
\end{aligned}$$

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We calculate ${}_tp_{65} = e^{-\int_0^t 0.0000023(1.12)^{65+t}\,dt} = e^{-0.0000023(1.12)^{65}\left(\frac{1.12^t-1}{\log(1.12)}\right)} = e^{-0.03210402(1.12^t-1)}$

$\ddot{a}_{65}^{(12)} = \frac{1}{12}\sum_{n=0}^{\infty} e^{-0.03210402(1.12^{\frac{n}{12}}-1)}(1.05^{-\frac{n}{12}} = 14.03364$

If the member withdraws at age x, then the salary is $45000(1.04)^{x-46}\left(\frac{1.04^{-1}+1.04^{-2}+1.04^{-3}}{3}\right)$, so with the COLA, the accrued pension has an annual value of $41626.3655\times 0.015\times 13(1.04)^{x-46}(1.02)^{65-x} = 11825.14209\left(\frac{1.04}{1.02}\right)^{x-46}$, so the value at age 65 is

$$165949.787\left(\frac{1.04}{1.02}\right)^{x-46}$$

. Discounting at 5% to the age of withdrawl and at 6% to the present day, gives a conditional present value of

$$165949.787\left(\frac{1.04\times 1.06}{1.02\times 1.05}\right)^{x-46}(1.05)^{46-65} = 65671.96589\,(1.029318394)^{x-46}$$

The EPV is therefore $65671.96589\,(1.029318394)^{x-46}\,{}_{65-x}p_{46+x}$

We have ${}_{65-x}p_{46+x} = e^{-0.0000023(1.12)^{65}\left(\frac{1-1.12^{x-65}}{\log(1.12)}\right)} = e^{-0.03210402(1-1.12^{x-65})}$ so the EPV is

$$65671.96589\,(1.029318394)^{x-46}\,e^{-0.03210402(1-1.12^{x-65})}$$

The probability that the member is still enrolled in the plan at age x is $e^{-\int_{46}^{x} e^{-0.07y}+0.0000023(1.12)^y\,dy} = e^{-\frac{(e^{-3.22}-e^{-0.07x})}{0.07}-0.00002029494998(1.12^x-1.12^{46})}$ the EPV of accrued pension benefits paid to early withdrawls is therefore

$$\int_{46}^{60} 65671.96\,(1.02932)^{x-46}\,e^{-0.03210(1-1.12^{x-65})}e^{-0.07x}e^{-\frac{(e^{-3.22}-e^{-0.07x})}{0.07}-0.000020295(1.12^x-1.12^{46})}$$
$$= 26792.25309$$

The probability he is still employed at age 60 is $e^{-\int_{46}^{60} e^{-0.07x}+0.0000023(1.12)^x\,dx} = e^{-\frac{e^{-3.22}-e^{-4.2}}{0.07}-0.0000023\frac{1.12^{60}-1.12^{46}}{\log(1.12)}} = 0.6900072247$

If this happens, then his final average salary is $45000\left(\frac{1.04^{17}+1.04^{18}+1.04^{19}}{3}\right) = 91208.4928$

He has probability 0.3 of retiring at age 60, in which case the expected value of the accrued pension is $91208.4928 \times 13 \times 0.015\ddot{a}_{60}^{(12)} = 272954.8931$. The EPV of pension benefits from retirements at age 60 is therefore $272954.8931 \times 0.3 \times 0.6900072247(1.06)^{-14} = 24991.00164$.

To simplify, we assume the remaining retirements, except at age 65 happen in the middle of their year. We get the following:

age	P(retire)	$\ddot{a}_x^{(12)}$	S_{Fin}	EPV(Pension Benefits)
60.5	0.007984369979	15.22710298	93014.77692	947.3478018
61.5	0.007517395163	14.98002832	96735.3680	860.9124779
62.5	0.007113475961	14.72279172	100604.7827	785.5584389
63.5	0.006768439094	14.45514241	104628.9740	720.0204573
64.5	0.006478882979	14.17684657	108814.1330	663.1949110
65	0.4471424941	14.03363874	110969.0775	44878.8153482
total				48855.84944

So the total EPV of accrued pension benefits is $48855.84944 + 26792.25309 + 24991.00164 = \$100,639.10$

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If the individual retires or withdraws at age x, then career average earnings time years of service is given by

$$180000\int_0^{x-38} 1.04^y\,dy = \frac{180000}{\log(1.04)}(1.04^{x-38}-1)$$

The life's current total pensionable earnings are therefore given by

$$\frac{180000}{\log(1.04)}(1.04^6-1) = \$1,217,658$$

so at an accrual rate of 4%, the annual pension is based on an annual pension rate of $0.04 \times 1217658 = \$48,706.33$. We will apply COLA of 2% to this, so if the life starts receiving the pension in t years, then the EPV of the accrued pension at that time is $48,706.33(1.02)^t\overline{a}_{44+t}$. Mortality is given by $\mu_x = 0.0000023(1.12)^x$, so ${}_tp_x = e^{-\int_x^{x+t} 0.0000023(1.12)^y\,dy} = e^{-\frac{0.0000023(1.12)^x}{\log(1.12)}(1.12^t-1)}$. This means that $\overline{a}_x = \int_0^\infty e^{\frac{0.0000023(1.12)^x}{\log(1.12)}(1.12^t-1)}1.05^{-t}\,dt$.

Numerically, we calculate $\overline{a}_{65} = 14.14552$, so the accrued pension benefits for a life who starts receiving pension benefits at age 65 are given by $14.14552 \times 48,706.33(1.02)^{21} = \$1,044,258$.

The probability density that the life withdraws at age $44 + t$ ($t < 16$) is

$$ {}_tp_{44} e^{-\int_0^t e^{-0.07(44+s)}\,ds} e^{-0.07(44+t)} = {}_tp_{44} e^{-e^{-3.08}\left(\frac{1-e^{-0.07t}}{0.07}\right)} e^{-0.07(44+t)} $$

so the EPV of withdrawl benefits is given by

$$ \begin{aligned} &1044258 \int_0^{16} {}_tp_{44} e^{-e^{-3.08}\left(\frac{1-e^{-0.07t}}{0.07}\right)} e^{-0.07(44+t)} {}_{21-t}p_{44+t}(1.06)^{-t}(1.05)^{t-21}\,dt \\ =&1044258\,{}_{21}p_{44} \int_0^{16} e^{-e^{-3.08}\left(\frac{1-e^{-0.07t}}{0.07}\right)} e^{-0.07(44+t)}(1.06)^{-t}(1.05)^{t-21}\,dt \end{aligned} $$

We can evaluate this expression numerically to get

$$ \int_0^{16} e^{-e^{-3.08}\left(\frac{1-e^{-0.07t}}{0.07}\right)} e^{-0.07(44+t)}(1.06)^{-t}(1.05)^{t-21}\,dt = 0.121347 $$

This means the EPV of accrued withdrawl benefits is given by

$$ 0.1471344 \times 1044258 e^{\frac{-0.0000023(1.12)^{44}}{\log(1.12)}(1.12^{21}-1)} = \$123,079.2 $$

The probability that the life is still employed at age 60 is

$$ {}_{16}p_{44} e^{-\frac{e^{-3.08}(1-e^{-1.12})}{0.07}} = e^{\frac{-0.0000023(1.12)^{44}}{\log(1.12)}(1.12^{16}-1)} e^{-\frac{e^{-3.08}(1-e^{-1.12})}{0.07}} = 0.632811 $$

This means the probability of retiring at age 60 is $0.632811 \times 0.3 = 0.1898433$. We calculate numerically $\overline{a}_{60} = 15.34081$, so the EPV of accrued benefits from retirement at age 60 is $48706.33 \times 15.34081(1.02)^{16}(1.06)^{-16} = 140572.8$. The probability of continuing to work past 60 is $0.632811 \times 0.7 = 0.4429677$. Given that an employee continues to work past 60, the probability that they retire at age 65 is ${}_5p_{60} e^{-0.06\times 5} = e^{-\frac{0.0000023(1.12)^{60}}{\log(1.12)}(1.12^5-1)} e^{-0.3} = 0.7306013$, so the probability that the life retires at age 65 is $0.7306013 \times 0.4429677 = 0.3236328$, so the EPV of accrued benefits from retirement at 65 is $0.3236328 \times 1044258(1.06)^{-21} = \$99,411.62$. For retirements between ages 60 and 65, conditional on being alive, retirements happen following an exponential distribution, so for an individual alive at age $60 + t$, who continued working past 60, the probability of not being retired is $e^{-0.06t}$, while the density of having retired at age $60 + s$ is $0.06e^{-0.06s}$. We can evaluate the EPV of retirement benefits for individuals aged 60–65 by integrating over all payment times t, and all possible retirement ages s as follows:

$$
\begin{aligned}
&\int_0^5 {}_tp_{60}(1.02)^{16+t}\int_0^t 0.06e^{-0.06s}(1.05)^{s-t}(1.06)^{-(16+s)}\,ds\,dt\\
&=\int_0^5 {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16+t}\int_0^t 0.06e^{-0.06s}\left(\frac{1.05}{1.06}\right)^{s-t}\,ds\,dt\\
&=\int_0^5 {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16+t}\left(\frac{1.05}{1.06}\right)^{-t}\frac{0.06}{0.06+\log\left(\frac{1.06}{1.05}\right)}\left(1-e^{-\left(0.06+\log\left(\frac{1.06}{1.05}\right)\right)t}\right)dt\\
&=\frac{0.06}{0.06+\log\left(\frac{1.06}{1.05}\right)}\int_0^5 {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16+t}\left(\frac{1.05}{1.06}\right)^{-t}\left(1-e^{-0.06t}\left(\frac{1.05}{1.06}\right)^{t}\right)dt\\
&=\frac{0.06}{0.06+\log\left(\frac{1.06}{1.05}\right)}\int_0^5 {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16+t}\left(\left(\frac{1.05}{1.06}\right)^{-t}-e^{-0.06t}\right)dt\\
&=0.2690331
\end{aligned}
$$

So the EPV of payments made between ages 60 and 65 is $0.2690331\times 48706.33\times 0.4429677=\$5,804.48$.

For payments made after age 60 to individuals who retire before age 65, we use a similar integral, but with the limits of the integral for s ranging from 0 to 5 (since we have already accounted for individuals who retire at age 65). We calculate

$$
\begin{aligned}
&\int_5^\infty {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16}\left(\frac{1.02}{1.05}\right)^{t}\int_0^5 0.06e^{-0.06s}\left(\frac{1.05}{1.06}\right)^{s}\,ds\,dt\\
&=\frac{0.06}{0.06+\log\left(\frac{1.06}{1.05}\right)}\left(1-e^{-\left(0.06+\log\left(\frac{1.06}{1.05}\right)\right)\times 5}\right)\int_5^\infty {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16}\left(\frac{1.02}{1.05}\right)^{t}dt\\
&=0.2534354\int_5^\infty {}_tp_{60}\left(\frac{1.02}{1.06}\right)^{16}\left(\frac{1.02}{1.05}\right)^{t}dt\\
&=1.517176
\end{aligned}
$$

The EPV of benefits paid after age 65 to individuals who retire between ages 60 and 65 is therefore $1.517176\times 48706.33\times 0.4429677=32733.57$.

The total EPV of all accrued benefits is therefore

$$123079.2+140572.8+99411.62+5804.48+32733.57=\$401,602$$

10.7 Funding the Benefits

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(a) Under the projected unit method, the final average salary is expected to be $47000\left(\frac{1.05^{18}+1.05^{19}+1.05^{20}}{3}\right)=118860.9184$

We have $\ddot{a}_{65}^{(12)}=13.95103541$. Therefore the EPV of accrued benefits for an individual who reaches retirement age is $118860.9184\times 0.02\times 26\times 13.95103541(1.04)^{-20}=393533.8372$.

The probability that this individual reaches retirement age is ${}_{20}p_{45}=e^{-0.0000076\left(\frac{1.1087^{65}-1.1087^{45}}{\log(1.1087)}\right)}=0.948743622$.

So the EPV of benefits is $0.948743622\times 393533.8372=373362.7181$.

After another year, the projected final average salary will still be 113200.8747, so the EPV conditional on surviving to retirement age will be $118860.9184 \times 0.02 \times 27 \times 13.95103541(1.04)^{-19} = 425016.5442$.

The probability of surviving to retirement age is $e^{-0.0000076\left(\frac{1.1087^{65}-1.1087^{46}}{\log(1.1087)}\right)} = 0.9495331634$. The EPV of benefits at the end of the year is therefore $0.9495331634 \times 425016.5442 = 403567.3037$.

The accumulated value of the reserves at the begining of the year is $373362.7181(1.04) = 388297.2268$, so the annual contribution is $403567.3037 - 388297.2268 = \$15,270.08$.

(b) Under the traditional unit method, the final average salary is $47000\left(\frac{1.05^{-2}+1.05^{-1}+1}{3}\right) = 44797.43008$

The value in the current year is therefore $44797.43008 \times 0.02 \times 26 \times 13.95103541(1.04)^{-20}{}_{20}p_{45} = 140716.4818$

If the member survives the year, the final average salary in one year's time is $47000\left(\frac{1.05^{-1}+1+1.05}{3}\right) = 47037.30159$ so EPV at the end of next year is $47037.30159 \times 0.02 \times 27 \times 13.95103541(1.04)^{-20}{}_{20}p_{45} = 159705.2861$.

The accumulated value of the assets funding the benefit at the start of the year is $140716.4818(1.04) = 146345.1411$, so the contribution is $159705.2861 - 146345.1411 = \$13,360.14$.

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Current reserve

Pension Benefits:

If the individual retires after t years, their final average salary is $87000\frac{(1.06)^{t-2}+(1.06)^{t-1}+(1.06)^{t})}{3} = 82168.39(1.06)^t$

The pension benefits are therefore $82168.39(1.06)^t \times 0.12\ddot{a}_{46+t:\overline{5}|}$ These need to be discounted by t years at $i = 0.05$. We then take the expectation over possible retirement times to calculate the total EPV of retirement benefits as

$$\int_{14}^{19} {}_tp_{46}^{00}\mu_{46+t}^{02} 82168.39(1.06)^t \times 0.12\ddot{a}_{46+t:\overline{5}|}(1.05)^{-t}\,dt$$

Deferred Pension Benefits:

Death Benefits:

Next year reserve

Pension Benefits:

Deferred Pension Benefits:

Death Benefits:

Benefits for exits during year

Deferred Pension Benefits:

Death Benefits:

The rate of exit (for ages below 60) is $\mu_x^{01} + \mu_x^{02} = 0.2e^{-0.04x} + 0.00000187(1.130)^x$ so the probability of the employee remaining employed at age x is

$$e^{-\int_{46}^{x} 0.2e^{-0.04t}+0.00000187(1.130)^t\,dt} = e^{-\left[\frac{0.00000187}{\log(1.130)}(1.130)^t - \frac{0.2}{0.04}e^{-0.04t}\right]_{46}^{x}} = e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)}$$

If the individual has retired, is at age x and has passed the guaranteed time of the pension, the value of the pension at that time is $R\sum_{i=0}^{\infty} {}_ip_x 1.05^{-i}$ where R is the regular pension payment. We have that

$${}_tp_x = e^{-0.00000187(1.13)^x\int_0^t e^{s\log(1.13)}\,ds} = e^{-0.00000187(1.13)^x\left(\frac{(1.13)^t-1}{\log(1.13)}\right)}$$

This gives that the value of the pension is

$$R\sum_{i=0}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^i-1}{\log(1.13)}\right)}1.05^{-i}$$

If the individual retires at age x, with pension value R then the present value of the pension at that time is

$$R\left(\ddot{a}_{\overline{5}|0.05}+\sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^i-1}{\log(1.13)}\right)}1.05^{-i}\right)$$

and the present value is

$$(1.05)^{-19}R\left(\ddot{a}_{\overline{5}|0.05}+\sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)}1.05^{-i}\right)$$

If the member exits at age x, then the final average salary is $87000(1.06)^{x-46}\left(\frac{1+(1.06)^{-1}+(1.06)^{-2}}{3}\right) = 82168.39(1.06)^{x-46}$.

If the member withdraws at age x, then the final average salary is also $82168.39(1.06)^{x-46}$, and with COLA, the eventual accrued pension benefit is $82168.39(1.06)^{x-46}(1.02)^{65-x} = 20400.75(1.039216)^x$ per 100 years of service.

Since the employee has 12 years of service, the accrued pension beneifit if he withdraws at age x is

$$2448.090(1.039216)^x$$

The overall expected pension benefit for an individual who withdraws is therefore

$$\begin{aligned}&\int_{46}^{60} 0.2e^{-0.04x}e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)}\times 2448.090(1.039216)^x\times\\&\qquad(1.05)^{-19}\left(\ddot{a}_{\overline{5}|0.05}+\sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)}1.05^{-i}\right)\,dx\\&=27393.66\end{aligned}$$

After another year, the expected pension benefit to an individual who withdraws will be

$$\begin{aligned}&\int_{47}^{60} 0.2e^{-0.04x}e^{-0.004229764\left((1.130)^{x-46}-1\right)-0.7940871\left(1-e^{-0.04(x-46)}\right)}\times 2448.090(1.039216)^x\times\\&\qquad(1.05)^{-19}\left(\ddot{a}_{\overline{5}|0.05}+\sum_{i=5}^{\infty} e^{-0.00000187(1.13)^x\left(\frac{(1.13)^{i+65-x}-1}{\log(1.13)}\right)}1.05^{-i}\right)\,dx\\&=25057.95\end{aligned}$$

In addition, annother year will have accrued, so the expected benefit is $\frac{13}{12}\times 25057.95 = 27146.11$.

(Valued at the present time, not at age 47.)

If the individual is still employed at age 60, and retires at age x, then the present value of the accrued pension is

$$0.12\times 87000(1+1.06^{-1}+1.06^{-2})1.06^{x-46}1.05^{46-x}\left(\ddot{a}_{\overline{5}|0.05}+\sum_{i=5}^{\infty}e^{-0.00000187(1.13)^x\left(\frac{(1.13)^i-1}{\log(1.13)}\right)}1.05^{-i}\right)$$

Denote this value by $P(x)$. We then have that the expected pension payments to individuals who retire are

$${}_{14}p_{46}^{(00)}\left(0.08P(60)+0.92\int_0^5 0.1e^{-0.1t-0.00000187(1.13)^{60}\left(\frac{(1.13)^t-1}{\log(1.13)}\right)}P(60+t)\,dt+{}_{19}p_{46}P(65)\right)=12053.66{}_{14}p_{46}^{(00)}$$

We have that

$${}_{14}p_{46}^{(00)}=e^{-0.004229764\left((1.130)^{14}-1\right)-0.7940871\left(1-e^{-0.04(14)}\right)}=0.6979008$$

So the expected payments for individuals who retire are $0.6979008\times 12053.66=\$8412.26$.

The expected payments conditional on being employed at the end of the year are

$$12053.66{}_{13}p_{47}^{(00)}=12053.66\times 0.7113312=8574.145$$

.

Multiplying by $\frac{13}{12}$ to account for the additional year of service, we get $\frac{13}{12}\times 8574.145=9288.66$.

Finally, conditional on the individual surviving for the year, the expected death benefits decrease by the expected death benefits for the year, which is

$$\int_0^1 0.00000187(1.13)^{46+t}e^{-\frac{0.00000187}{\log(1.13)}(1.13)^{46}(1.13^t-1)-5e^{-1.84}(1-e^{-0.04t})}\times 3\times 87000(1.06^t)\,dt=145.47$$

The increase in the present value of expected benefits is therefore

$$27146.11-27393.66+9288.66-8412.26-145.47=\$483.38$$

The employee contribution is 4% of 87000, which is \$3,480.

Retiree Health Benefits

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The proportion of benefits accrued depends on the age of retirement. For an individual retiring at age 61, their benefits should be accrued over a longer period than for an individual retiring aged 60. A more conservative approach that is sometimes used is to accrue benefits for all retirees up to the minimum retirement age, then for employees above that age, the benefits are fully paid up, so no normal contribution is necessary.

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We calculate $i^*=\frac{1.06}{1.02\times 1.05}-1=-0.01027077$. If the individual retires aged 60, they will have 14 years of service, and the premium at the time of retirement will be $984(1.05)^7=1384.59$, the EPV at age 60 of the benefits will be $984(1.05)^7\times 42.26993=58526.39$. The current accrued EPV will be $58526.39(1.06)^{-7}\frac{7}{14}=\$19,461.70$ while the accrued EPV after another year will be $58526.39(1.06)^{-6}\frac{8}{14}=\$23,576.45$. The normal contribution is therefore $4114.75(1.06)^{-1} = \$3,881.85$. We calculate similarly for other retirement ages in the following table:

x_r	EPV at x_r	YoS at x_r	Current Accrued EPV	Accrued EPV in 1 year	Normal Contribution	Probability	Expected Normal Contribution
60	58526.39	14	19461.70	23576.45	3881.85	0.185	718.14
60.5	59038.60	14.5	18410.77	22303.34	3672.23	0.032	117.51
61.5	60046.08	15.5	16525.37	20019.30	3296.16	0.039	128.55
62.5	61038.36	16.5	14887.14	18034.71	2969.40	0.044	130.65
63.5	62012.21	17.5	13453.20	16297.59	2683.39	0.051	136.85
64.5	62965.74	18.5	12190.27	14767.64	2431.48	0.016	38.90
65	63430.00	19	11613.65	14069.11	2316.47	0.580	1343.55

The total normal contribution for the year is therefore
$718.14 + 117.51 + 128.55 + 130.65 + 136.85 + 38.90 + 1343.55 = \$2,614.17$.

12.3 Profit Testing a Term Insurance Policy

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See slide.

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See slide.

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See slide.

12.5 Profit Measures

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See slide.

12.6 Using the Profit Test to Calculate the Premium

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At a risk discount rate of 10%, the NPV is

$$P\left((1.04)(1.1)^{-1} + 0.9996\left(\frac{1.1^{-1} - 1.1^{-10}}{0.1}\right)\right) - (160 + 56.34(1.1)^{-1} + 59.24(1.1)^{-2} + 62.50(1.1)^{-3} + 66.13(1.1)^{-4}$$
$$+69.93(1.1)^{-5} + 74.11(1.1)^{-6} + 78.65(1.1)^{-7} + 83.38(1.1)^{-8} + 88.83(1.1)^{-9} + 94.47(1.1)^{-10}) = 6.178837P - 590.908$$

The premium should be chosen to make this equal to zero. That is

$$\begin{aligned} 6.178837P - 590.908 &= 0 \\ 6.178837P &= 590.908 \\ P &= \$95.63 \end{aligned}$$

12.7 Using the Profit Test to Calculate Reserves

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Recalling Question 65, with a premium of \$90, The net outflows without reserves are all positive except for the final year, where the net outflow is -4.61. To correct for this, the reserve R must satisfy $1.04R = 4.61$. This means $R = \$4.43$. This reserve is a negative cash flow at the end of year 9, so the net cash flow from year 9 is $1.03 - 4.43 = -3.40$. To prevent this negative cash flow, the reserve needs to satisfy $1.04R = 3.40$, so $R = 3.27$. This is a negative cash flow at the end of year 8, so the net cash-flow in that year is $6.48 - 3.27 = 3.21$, which is positive, so reserves of \$3.27 in year 9 and \$4.43 in year 10 are needed.

12.8 Profit Testing for Multiple-State Models

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Recall from Question 6 that the probabilities of the life being in each state are:

t	${}_tp_{37}^{00}$	${}_tp_{37}^{01}$	${}_tp_{37}^{02}$
0	1	0	0
1	0.99812	0.000375	0.001505
2	0.99617	0.000750	0.003083
3	0.99414	0.001127	0.004736
4	0.99203	0.001505	0.006464
5	0.98985	0.001884	0.008271
6	0.98758	0.002263	0.010156
7	0.98523	0.002644	0.012123
8	0.98280	0.003025	0.014171
9	0.98029	0.003407	0.016303
10	0.97769	0.003790	0.018519

If the life is in the healthy state at the start of year i, and is then aged x, the probability of being sick at the end is

$$e^{0.000001\begin{pmatrix} -\left(1881+\frac{1}{3}+77x+x^2\right) & 375+2x & 1506+\frac{1}{3}+75x+x^2 \\ 67.5+x & -(342.5+3x) & 275+2x \\ 0 & 0 & 0 \end{pmatrix}}$$

Probabilities of being sick or dead at the end of each year for a life alive at start of year are

t	p_{37+t}^{00}	p_{37+t}^{01}	p_{37+t}^{02}	p_{37+t}^{10}	p_{37+t}^{11}	p_{37+t}^{12}
0	0.9981204	0.0003745833	0.001504969	0.00006742499	0.9996576	0.0002750037
1	0.9980426	0.0003765658	0.001580836	0.00006842111	0.9996546	0.0002770063
2	0.9979628	0.0003785478	0.001658694	0.00006941708	0.9996516	0.0002790090
3	0.9978809	0.0003805293	0.001738542	0.00007041290	0.9996486	0.0002810119
4	0.9977971	0.0003825102	0.001820380	0.00007140856	0.9996456	0.0002830149
5	0.9977113	0.0003844905	0.001904206	0.00007240405	0.9996426	0.0002850181
6	0.9976235	0.0003864703	0.001990020	0.00007339939	0.9996396	0.0002870214
7	0.9975337	0.0003884495	0.002077823	0.00007439457	0.9996366	0.0002890249
8	0.9974420	0.0003904281	0.002167613	0.00007538957	0.9996336	0.0002910285
9	0.9973482	0.0003924062	0.002259390	0.00007638440	0.9996306	0.0002930323

t	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Net Cash Flow
0		200				−200
1	489.45		34.26	29.96666	300.9938	192.75108
2	489.45		34.26	30.12527	316.1673	177.41896
3	489.45		34.26	30.28383	331.7389	161.68881
4	489.45		34.26	30.44234	347.7084	145.56071
5	489.45		34.26	30.60081	364.0759	129.03478
6	489.45		34.26	30.75924	380.8412	112.11110
7	489.45		34.26	30.91762	398.0041	94.78978
8	489.45		34.26	31.07596	415.5646	77.07092
9	489.45		34.26	31.23425	433.5226	58.95464
10	489.45		34.26	31.39249	451.8780	40.44103

t	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Net Cash Flow
0		200				−200
1	489.45		34.26	79972.61	55.00074	−79503.89
2	489.45		34.26	79972.37	55.40126	−79504.06
3	489.45		34.26	79972.13	55.80181	−79504.22
4	489.45		34.26	79971.89	56.20238	−79504.38
5	489.45		34.26	79971.65	56.60299	−79504.54
6	489.45		34.26	79971.41	57.00362	−79504.70
7	489.45		34.26	79971.17	57.40429	−79504.86
8	489.45		34.26	79970.93	57.80498	−79505.02
9	489.45		34.26	79970.69	58.20571	−79505.18
10	489.45		34.26	79970.45	58.60646	−79505.34

We see that expected cash flows are all negative if the life starts the year in the sick state, and positive if the life starts the year in the healthy state. We need to calculate separate reserves for each state in the usual way by working backwards.

For the 10th year, if the life is in the sick state, the expected net cash flow is $-\$79,505.34$, so the reserve needed is $79505.34(1.07)^{-1} = 74304.06$.

For a life in the sick state at the start of year 9, the probability that the life is in the sick state at the end of year 9 is 0.9996306, so the expected reserves needed are $0.9996306 \times 74304.06 = \$74276.61$. For a life in the healthy state at the start of year 9, the probability of being in the sick state at the end is 0.0003924, so the additional expected cashflow is $-0.0003924 \times 74304.06 = -\$29.15691$, making the total expected net cash flow for that year \$29.80.

We proceed up the table in this way.

t	Reserve	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Expected Reserve	Net Cash Flow
2	530748.32	489.45		34.26	79972.37	55.40126	488396.64	−567900.70
3	488566.86	489.45		34.26	79972.13	55.80181	443262.32	−522766.54
4	443418.14	489.45		34.26	79971.89	56.20238	394953.03	−474457.41
5	395093.05	489.45		34.26	79971.65	56.60299	343245.02	−422749.56
6	343367.74	489.45		34.26	79971.41	57.00362	269064.28	−348568.98
7	269161.28	489.45		34.26	79971.17	57.40429	208497.72	−288002.58
8	208573.51	489.45		34.26	79970.93	57.80498	143668.64	−223173.66
9	143721.30	489.45		34.26	79970.69	58.20571	74276.61	−153781.79
10	74276.61	489.45		5235.55	79970.45	58.60646	0	−79505.34

There is no need to calculate the reserves for the first year, since we know that the life is in the healthy state at the start of year 1.

Knowing the reserves needed if the life is in the sick state, we can calculate the reserves as extra expenses in the healthy state:

t	Reserve	Premium	Exp	Interest	Expected Disability Benefit	Expected Death Benefit	Expected Reserve Sick	Expected Reserve Healthy	Net Cash Flow
0			200					21.65	−221.65
1	21.65	489.45		34.26	29.96666	300.9938	198.81	17.11	−23.17
2	17.14	489.45		34.26	30.12527	316.1673	183.98	11.78	−18.34
3	11.80	489.45		34.26	30.28383	331.7389	167.85	6.47	−12.63
4	6.49	489.45		34.26	30.44234	347.7084	150.34	2.15	−6.94
5	2.16	489.45		34.26	30.60081	364.0759	131.34		−2.31
6		489.45		34.26	30.75924	380.8412	103.49		8.62
7		489.45		34.26	30.91762	398.0041	80.61		14.18
8		489.45		34.26	31.07596	415.5646	55.83		21.24
9		489.45		34.26	31.23425	433.5226	29.00		29.95
10		489.45		34.26	31.39249	451.8780			40.44

The resulting profit signature is then

$$-221.65 \times 1 = -221.65$$
$$0 \times 1 = 0$$
$$0 \times 0.99812 = 0$$
$$0 \times 0.99617 = 0$$
$$0 \times 0.99414 = 0$$
$$0 \times 0.99203 = 0$$
$$8.62 \times 0.98985 = 8.53$$
$$14.18 \times 0.98758 = 14.00$$
$$21.24 \times 0.98523 = 20.93$$
$$29.95 \times 0.98280 = 29.43$$
$$40.44 \times 0.98029 = 39.64$$