ACSC/STAT 4720, Life Contingencies II
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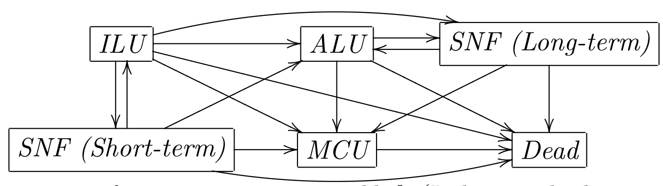
Homework Sheet 1

Model Solutions

Basic Questions

1. *An CCRC is developing a model for its care costs. The community has four levels of care: Independent Living Unit, Assisted Living Unit, Skilled Nursing Facility, and Memory Care Unit. The transition diagram is shown below:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) ILU–SNF (long-term)– ALU–Dead

This is possible.

(ii) ILU–ALU–SNF (short-term)–ALU

This is impossible, because the transition ALU–SNF (short-term) is not possible.

(iii) ILU–MCU–ALU–Dead

This is impossible, because the transition MCU–ALU is not possible.

(iv) ILU–SNF (short-term)–ILU–ALU

This is possible.

(v) ILU–MCU–SNF (long-term)–Dead

This is impossible, because the transition MCU–SNF (long-term) is not possible.

2. *Consider a permanent disability model with transition intensities*

$$\mu_x^{01} = 0.001 + 0.000003x$$
$$\mu_x^{02} = 0.001 + 0.000004x$$
$$\mu_x^{12} = 0.004 + 0.000002x$$

where State 0 is healthy, State 1 is permanently disabled and State 2 is dead.

(a) Calculate the probability that a healthy individual aged 27 is still healthy at age 44.

This is given by

$$e^{-\int_{27}^{44} 0.002+0.000007x\,dx} = e^{-\left[0.002x+0.0000035x^2\right]_{27}^{44}} = e^{-\left(0.002\times 17+0.0000035(44^2-27^2)\right)} = e^{-(0.034+0.0042245)} = 0.962496836095$$

(b) Calculate the probability that a healthy individual aged 33 is dead by age 56.

There are two ways the individual can be dead — dying directly from the healthy state, and becoming critically ill first. If the life becomes permanently disabled at age a, the probability that the life is still alive at age 56 is given by

$$e^{-\int_{a}^{56} 0.004+0.000002x\,dx} = e^{-\left(0.004(56-a)+0.000002(56^2-a^2)\right)}$$

The probability density of becoming permanently disabled at age a is

$$(0.001+0.000003a)e^{-\int_{33}^{a} 0.002+0.000007x\,dx} = (0.001+0.000003a)e^{-\left(0.002(a-33)+0.000007(a^2-33^2)\right)}$$

This means the probability that the life is permanently disabled at age 56 is

$$\begin{aligned}
&\int_{33}^{56}(0.001+0.000003a)e^{-\left(0.002(a-33)+0.000007(a^2-33^2)\right)}e^{-\left(0.004(56-a)+0.000002(56^2-a^2)\right)}\,da\\
=&\int_{33}^{56}(0.001+0.000003a)e^{0.066-0.224+0.002a+0.007623-0.006272-0.000005a^2}\,da\\
=&\int_{33}^{56}(0.001+0.000003a)e^{-0.000005(a^2-400a)-0.156649}\,da\\
=&\int_{33}^{56}(0.001+0.000003a)e^{-0.000005(a-200)^2-0.243351}\,da\\
=&e^{-0.243351}\int_{33}^{56}(0.000003(a-200)+0.0016)e^{-0.000005(a-200)^2}\,da\\
=&0.000003e^{-0.243351}\int_{33}^{56}(a-200)e^{-0.000005(a-200)^2}\,da+0.0016e^{-0.243351}\int_{33}^{56}e^{-0.000005(a-200)^2}\,da\\
=&0.000003e^{-0.243351}\left[-100000e^{-0.000005(a-200)^2}\right]_{33}^{56}+100\sqrt{20\pi}0.0016e^{-0.243351}\int_{33}^{56}\frac{1}{100\sqrt{20\pi}}e^{-0.000005(a-200)^2}\,da\\
=&0.3e^{-0.243351}\left(e^{-0.000005(33-200)^2}-e^{-0.000005(56-200)^2}\right)+0.16e^{-0.243351}\sqrt{20\pi}\left(\Phi\left(\frac{56-200}{100\sqrt{10}}\right)-\Phi\left(\frac{33-200}{100\sqrt{10}}\right)\right)\\
&=0.0123657
\end{aligned}$$

The probability that the life is healthy is given by

$$e^{-\int_{33}^{56} 0.002+0.000007x\,dx} = e^{-\left[0.002x+0.0000035x^2\right]_{33}^{56}} = e^{-\left(0.002\times 23+0.0000035(56^2-33^2)\right)} = e^{-(0.046+0.0071645)} = 0.948224016801$$

The probability that the life is dead is therefore $1-0.948224016801-0.0123657 = 0.039410283$.

3. *Under a disability income model with transition intensities*

$$\begin{aligned}\mu_x^{01} &= 0.002\\ \mu_x^{10} &= 0.004\\ \mu_x^{02} &= 0.001\\ \mu_x^{12} &= 0.006\end{aligned}$$

calculate the probability that a healthy individual has some period of disability within the next 6 years. [State 0 is healthy, State 1 is sick and State 2 is dead.]

The probability that an individual remains healthy for the entire 6 years is $e^{-0.003\times 6} = e^{-0.018} = 0.982161032358$. The probability that the individual dies directly from the healthy state without ever becoming disabled is

$$\int_0^6 0.001e^{-0.003t}\,dt = 0.001\left[-\frac{e^{-0.003t}}{0.003}\right]_0^6 = \frac{1}{3}(1-e^{-0.018}) = 0.00594632254733$$

The probability that the individual has some period of disability is therefore $1-0.982161032358-0.00594632254733 = 0.0118926450947$.

4. *Under a critical illness model with transition intensities at age x given by:*

$$\begin{aligned}\mu_x^{01} &= 0.001 + 0.000006x\\ \mu_x^{02} &= 0.002\\ \mu_x^{12} &= 0.12\end{aligned}$$

calculate the premium for a whole life policy sold to a life aged 35 with premiums payable continuously while the life is in the healthy state, which pays a death benefit of \$130,000 upon entry into state 2, and a benefit of \$120,000 upon entry into state 1, sold to a life in the healthy state (state 0). The interest rate is $\delta = 0.04$ [State 0 is healthy, State 1 is sick and State 2 is dead.]

We have ${}_tp_{35}^{00} = e^{-\int_0^t 0.003+0.000006(35+t)\,dt} = e^{-\int_0^t 0.003+0.000006(35+t)\,dt} = e^{-0.00321t-0.000003t^2}$. We therefore calculate

$$\begin{aligned}\overline{a}_{\overline{10|}}^{00} &= \int_0^\infty e^{-0.04t}e^{-0.00321t-0.000003t^2}\,dt\\ &= \int_0^\infty e^{-0.000003(t+7201.66667)^2+155.592008477}\,dt\\ &= \sqrt{\frac{\pi}{0.000003}}e^{155.592008477}\left(1-\Phi\left(\sqrt{0.000006}\times 7201.66667\right)\right)\\ &= 23.06913\end{aligned}$$

Next, we can calculate

$$
\begin{aligned}
\overline{A}_{35}^{01} &= \int_0^\infty (0.001 + 0.000006(35+t))e^{-0.04t}e^{-0.00321t-0.000003t^2}\,dt \\
&= \int_0^\infty (0.00142 + 0.000006t)e^{-0.000003(t+7201.66667)^2+155.592008477}\,dt \\
&= \int_0^\infty (0.000006(t+7201.66667) - 0.04179)\,e^{-0.000003(t+7201.66667)^2+155.592008477}\,dt \\
&= e^{155.592008477} \times 0.000006 \left[-\frac{e^{-0.000003(t+7201.66667)^2}}{0.000006}\right]_0^\infty - 0.04179\overline{a}_{35}^{00} \\
&= 1 - 0.04179 \times 23.06913 \\
&= 0.0359410573
\end{aligned}
$$

$$
\begin{aligned}
\overline{A}_{35}^{02} = &\int_0^\infty 0.002e^{-0.04t}e^{-0.00321t-0.000003t^2}\,dt \\
&+ \int_0^\infty (0.001 + 0.000006(35+t))e^{-0.04t}e^{-0.00321t-0.000003t^2} \int_t^\infty e^{-0.04(s-t)}e^{-0.12(s-t)}(0.12)\,ds\,dt \\
= &\ 0.002 \times 23.06913 + \int_0^{10} (0.001 + 0.000006(35+t))e^{-0.04t}e^{-0.00321t-0.000003t^2}\left[-e^{-0.16(s-t)}\frac{0.12}{0.16}\right]_t^\infty dt \\
= &\ 0.016238102 + \frac{0.12}{0.16} \times 0.0359410573 \\
= &\ 0.043193894975
\end{aligned}
$$

The annual rate of premium is therefore $\frac{0.0359410573\times120000+0.043193894975\times130000}{23.06913} = \$430.36$

5. *An insurer offers a life insurance policy with an additional benefit for accidental death. The possible exits from this policy are surrender, death (accident) and death (other). The transition intensities are*

$$
\begin{aligned}
\mu_x^{01} &= 0.002 + 0.000001x \\
\mu_x^{03} &= 0.001 + 0.000006x \\
\mu_x^{02} &= 0.004 - 0.000002x
\end{aligned}
$$

Calculate the probability that an individual aged 34 dies in an accident before age 72. [State 0 is in force, State 1 is surrender, State 2 is death (accident) and State 3 is death (other).]

We have

$$
{}_tp_{34}^{00} = e^{-\int_0^t 0.007+0.000005(34+t)\,dt} = e^{-0.007306t-0.0000025t^2}
$$

This gives us

$$\begin{aligned}
{}_{38}p_{34}^{02} &= \int_0^{38} (0.004 - 0.000002(34+t))e^{-0.007306t - 0.0000025t^2} \\
&= \int_0^{38} -0.000002(t-1966)e^{-0.0000025(t^2+2922.4t)} \\
&= \int_0^{38} -0.000002(t-1966)e^{-0.0000025(t+1461.2)^2+5.3377636} \\
&= e^{5.3377636}\int_0^{38} (0.0068544 - 0.000002(t+1461.2))\, e^{-0.0000025(t+1461.2)^2} \\
&= 0.0068544\sqrt{\frac{\pi}{0.0000025}}e^{5.3377636}\left(\Phi\left(\sqrt{0.000005}\times 1499.2\right) - \Phi\left(\sqrt{0.000005}\times 1461.2\right)\right) \\
&\qquad - \frac{0.000002}{0.000005}e^{5.3377636}\left[-e^{-0.0000025(t+1461.2)^2}\right]_0^{38} \\
&= 0.2271822 - 0.4\left(1 - e^{-0.281238}\right) \\
&= 0.129121664048
\end{aligned}$$

Standard Questions

6. *An insurance company is developing a new model for transition intensities in a disability income model. Under these transition intensities it calculates*

$$\begin{aligned}
\overline{A}_{34}^{02} &= 0.217118 & \overline{A}_{49}^{02} &= 0.25344 & \overline{A}_{49}^{12} &= 0.0777432 \\
\overline{a}_{34}^{00} &= 12.0453 & \overline{a}_{49}^{00} &= 11.2778 & \overline{a}_{49}^{10} &= 0.033278 \\
{}_{15}p_{34}^{00} &= 0.723952 & {}_{15}p_{34}^{01} &= 0.0633742 & \delta &= 0.05
\end{aligned}$$

Calculate the premium for a 15-year policy for a life aged 34, with continuous premiums payable while in the healthy state, which pays a continuous benefit while in the sick state, at a rate of \$120,000 per year, and pays a death benefit of \$700,000 immediately upon death. [Hint: to calculate $\overline{a}_x^{01}$, consider how to extend the equation $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$ to the multiple state case by combining states 0 and 1.]

By combining the states 0 and 1, we get that $\overline{a}x^{00} + \overline{a}_x^{01} = \frac{1-\overline{A}_x^{02}}{\delta}$. This gives us

$$\overline{a}_x^{01} = \frac{1-\overline{A}_x^{02}}{\delta} - \overline{a}_x^{00}$$

so

$$\overline{a}_{34}^{01} = \frac{1-\overline{A}_{34}^{02}}{\delta} - \overline{a}_{34}^{00} = \frac{1-0.217118}{0.05} - 12.0453 = 3.61234$$
$$\overline{a}_{49}^{01} = \frac{1-\overline{A}_{49}^{02}}{\delta} - \overline{a}_{49}^{00} = \frac{1-0.25344}{0.05} - 11.2778 = 3.6534$$
$$\overline{a}_{49}^{11} = \frac{1-\overline{A}_{49}^{12}}{\delta} - \overline{a}_{49}^{10} = \frac{1-0.0777432}{0.05} - 0.033278 = 18.411858$$

The premium is

$$P = \frac{120000\overline{a}_{34:\overline{15}|}^{01} + 700000A_{34:\overline{15}|}^{02}}{\overline{a}_{34:\overline{15}|}^{00}}$$

We compute

$$\begin{aligned}
\overline{a}_{34:\overline{15}|}^{00} &= \overline{a}_{34}^{00} - {}_{15}p_{34}^{00}e^{-15\delta}\overline{a}_{49}^{00} - {}_{15}p_{34}^{01}e^{-15\delta}\overline{a}_{49}^{10} \\
&= 12.0453 - 0.723952e^{-0.75} \times 11.2778 - 0.0633742e^{-0.75} \times 0.033278 = 8.18762651481 \\
\overline{a}_{34:\overline{15}|}^{01} &= \overline{a}_{34}^{01} - {}_{15}p_{34}^{00}e^{-15\delta}\overline{a}_{49}^{01} - {}_{15}p_{34}^{01}e^{-15\delta}\overline{a}_{49}^{11} \\
&= 3.61234 - 0.723952e^{-0.75} \times 3.6534 - 0.0633742e^{-0.75} \times 18.411858 = 1.81180954268 \\
A_{34:\overline{15}|}^{02} &= A_{34}^{02} - {}_{15}p_{34}^{00}e^{-15\delta}A_{49}^{02} - {}_{15}p_{34}^{01}e^{-15\delta}A_{49}^{12} \\
&= 0.217118 - 0.723952e^{-0.75}0.25344 - 0.0633742e^{-0.75}0.0777432 = 0.128121634149
\end{aligned}$$

The annual rate of premium is therefore

$$P = \frac{120000 \times 1.81180954268 + 700000 \times 0.128121634149}{8.18762651481} = \$37508.10$$