ACSC/STAT 4720, Life Contingencies II
FALL 2021
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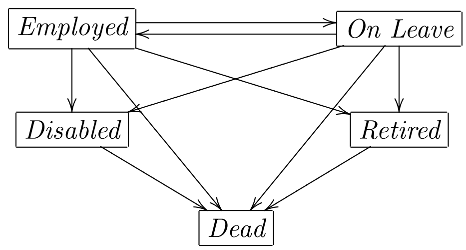
Sample Midterm Examination
Model Solutions

This Sample examination has more questions than the actual midterm, in order to cover a wider range of questions. Estimated times are provided after each question to help your preparation.

1. *An insurance company is considering a new policy. The policy includes states with the following state diagram:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) Employed—Disabled—Retired—Dead

This is not possible, since it is impossible to transition from *Disabled* to *Retired*

(ii) Employed—On Leave—Retired—Dead

This is possible.

(iii) Employed—Retired—On Leave—Dead

This is not possible, since it is impossible to transition from *Retired* to *On Leave*.

(iv) Employed—On Leave—Employed—Retired—Dead

This is possible.

2. *Consider a permanent disability model with transition intensities*

$$\mu_x^{01} = 0.002 + 0.000005x$$
$$\mu_x^{02} = 0.001 + 0.0000004x^2$$
$$\mu_x^{12} = 0.003 + 0.000004x$$

where State 0 is healthy, State 1 is permanently disabled and State 2 is dead. Write down an expression for the probability that an individual aged 29 is alive but permanently disabled at age 56. [You do not need to evaluate the expression, but should perform basic simplifications on it.]

The probability that the individual is in State 1 after 27 years is

$$
\begin{aligned}
{}_{27}p_{29}^{01} &= \int_0^{27} {}_tp_{29}^{\overline{00}}\mu_{29+t}^{01}{}_{27-t}p_{29+t}^{\overline{11}}\,dt \\
&= \int_0^{27} e^{-\int_0^t (0.003+0.000005(29+s)+0.0000004(29+s)^2)\,ds}(0.002+0.000005(29+t))e^{-\int_t^{27}(0.003+0.000004(29+s))\,ds}\,dt \\
&= \int_0^{27} e^{-\int_0^t (0.003+0.000145+0.000005s+0.0003364+0.0000232s+0.0000004s^2)\,ds}(0.002+0.000005(29+t))e^{-\int_t^{27}(0.003+\cdots} \\
&= \int_0^{27} e^{-\int_0^t (0.0034814+0.0000282s+0.0000004s^2)\,ds}(0.002+0.000005(29+t))e^{-\int_t^{27}(0.003116+0.000004s)\,ds}\,dt \\
&= \int_0^{27} e^{-(0.0034814t+0.0000141t^2+\frac{0.0000004}{3}t^3)\,ds}(0.002+0.000005(29+t))e^{-(0.003116(27-t)+0.000002(27^2-t^2))}\,dt \\
&= \int_0^{27} e^{-(0.0034814t+0.0000141t^2+\frac{0.0000004}{3}t^3)\,ds}(0.002+0.000005(29+t))e^{-(0.003116(27-t)+0.000002(27^2-t^2))}\,dt \\
&= \int_0^{27} e^{-(0.0034814t+0.0000141t^2+\frac{0.0000004}{3}t^3)\,ds}(0.002+0.000005(29+t))e^{-(0.08559-0.003116t-0.000002t^2)}\,dt \\
&= \int_0^{27} (0.002+0.000005(29+t))e^{-(0.08559+0.0003654t+0.0000121t^2+\frac{0.0000004}{3}t^3)\,ds}\,dt
\end{aligned}
$$

3. A disability income model has transition intensities

$$
\begin{aligned}
\mu_x^{01} &= 0.002 \\
\mu_x^{10} &= 0.001 \\
\mu_x^{02} &= 0.002 \\
\mu_x^{12} &= 0.004
\end{aligned}
$$

State 0 is healthy, State 1 is sick and State 2 is dead. Three actuaries calculate different values for the transition probabilities and benefit values. Which one has calculated plausible values? Justify your answer by explaining what is impossible about the values calculated by the other two actuaries.

Value	*Actuary I*	*Actuary II*	*Actuary III*
${}_2p_{37}^{(00)}$	*0.992036*	*0.992036*	*0.992036*
${}_2p_{37}^{(01)}$	*0.003960*	*0.003968*	*0.003964*
${}_4p_{37}^{(01)}$	*0.007857*	*0.007857*	*0.007857*
${}_4p_{37}^{(02)}$	*0.015857*	*0.008000*	*0.008000*
${}_4p_{37}^{(12)}$	*0.008000*	*0.015857*	*0.015857*
${}_2p_{39}^{(01)}$	*0.003960*	*0.003968*	*0.003964*
${}_2p_{39}^{(11)}$	*0.992054*	*0.992054*	*0.990054*

We have that ${}_4p_{37}^{(01)} ={}_2 p_{37}^{(00)} \times_2 p_{39}^{(01)} +_2 p_{37}^{(01)} \times_2 p_{39}^{(11)}$. For the numbers in the table, this gives

$$0.007857 = 0.992036 \times 0.003960 + 0.003960 \times 0.992054 = 0.007857$$
$$0.007857 = 0.992036 \times 0.003968 + 0.003968 \times 0.992054 = 0.007873$$
$$0.007857 = 0.992036 \times 0.003964 + 0.003964 \times 0.990054 = 0.007857$$

So the second actuary's calculations cannot be right. Furthermore, since $\mu_x^{02} < \mu_x^{12}$ for all x, we should have ${}_4p_{37}^{02} <_4 p_{37}^{12}$, which rules out the first actuary's calculations. This means that only Actuary III's calculations might be correct. [Indeed these are the correct values.]

4. *A disability income model has the following four states:*

State	*Meaning*
0	*Healthy*
1	*Sick*
2	*Accidental Death*
3	*Other Death*

The transition intensities are:

$$\begin{aligned}
\mu_x^{01} &= 0.001\\
\mu_x^{02} &= 0.002\\
\mu_x^{03} &= 0.001\\
\mu_x^{10} &= 0.002\\
\mu_x^{12} &= 0.001\\
\mu_x^{13} &= 0.003
\end{aligned}$$

You calculate that the probability that the life is healthy t years from the start of the policy is $0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t}$, *and the probability that the life is sick t years from the start of the policy is* $0.2886752e^{-0.003267949t} - 0.2886752e^{-0.006732051t}$.

Calculate the premium for a 5-year policy with premiums payable continuously while the life is in the healthy state, which pays no benefits while the life is in the sick state, but pays a benefit of \$200,000 in the event of accidental death and a benefit of \$100,000 in the event of other death. The interest rate is $\delta = 0.03$.

We calculate

$$\begin{aligned}
\overline{a}_{x:\overline{5}|} &= \int_0^5 e^{-0.03t}(0.2113249e^{-0.006732051t} + 0.7886751e^{-0.003267949t})\,dt\\
&= \int_0^5 0.2113249e^{-0.036732051t}\,dt + \int_0^5 0.7886751e^{-0.033267949t})\,dt\\
&= \frac{0.2113249}{0.036732051}(1 - e^{-0.036732051\times 5}) + \frac{0.7886751}{0.033267949}(1 - e^{-0.033267949\times 5})\\
&= 4.598130
\end{aligned}$$

The EPV of the benefits to lives who die accidentally from State 0 are given by

$$\begin{aligned}
&200000\int_0^5 0.002(0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=400\int_0^5 (0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=400\times 4.598130\\
&=1839.25
\end{aligned}$$

The EPV of the benefits to lives who die otherwise from State 0 are given by

$$\begin{aligned}
&100000\int_0^5 0.001(0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=100\int_0^5 (0.2113249e^{-0.006732051t}+0.7886751e^{-0.003267949t})e^{-0.03t}\,dt\\
&=100\times 4.598130\\
&=459.81
\end{aligned}$$

The EPV of the benefits to lives who die accidentally from State 1 are given by

$$\begin{aligned}
&200000\int_0^5 0.001(0.2886752e^{-0.003267949t}-0.2886752e^{-0.006732051t})e^{-0.03t}\,dt\\
&=200\int_0^5 0.2886752e^{-0.033267949t}-0.2886752e^{-0.036732051t})\,dt\\
&=\frac{57.73504}{0.033267949}(1-e^{-5\times 0.033267949})-\frac{57.73504}{0.036732051}(1-e^{-5\times 0.036732051})\\
&=2.226627
\end{aligned}$$

The EPV of the benefits to lives who die otherwise from State 1 are given by

$$\begin{aligned}
&100000\int_0^5 0.003(0.2886752e^{-0.003267949t}-0.2886752e^{-0.006732051t})e^{-0.03t}\,dt\\
&=300\int_0^5 0.2886752e^{-0.033267949t}-0.2886752e^{-0.036732051t})\,dt\\
&=3.339940
\end{aligned}$$

The total EPV of benefits is therefore $1839.25+459.81+2.23+3.34=\$2,304.63$. The annual rate of premium is $\frac{2304.63}{4.598130}=\$501.21$.

5. *Under a certain model for transition intensities in a critical illness model, with the following transition diagram:*

$$\begin{tikzcd} \boxed{\textit{0 Healthy}} \arrow[r] \arrow[dr] & \boxed{\textit{1 Critically Ill}} \arrow[d] \\ & \boxed{\textit{2 Dead}} \end{tikzcd}$$

you calculate:

$$\begin{aligned}
{}_5p_{41}^{00} &= 0.866102 & {}_5p_{41}^{01} &= 0.0542667 & {}_5p_{41}^{02} &= 0.0796309 \\
\overline{a}_{41}^{00} &= 13.5501 & \overline{a}_{41}^{01} &= 2.48302 & \overline{a}_{41}^{02} &= 8.96688 \\
\overline{a}_{46}^{0,0} &= 13.1355 & \overline{a}_{46}^{0,1} &= 2.49464 & \overline{a}_{46}^{0,2} &= 9.36984 \\
\overline{a}_{46}^{1,1} &= 13.2984 & \overline{a}_{46}^{1,2} &= 11.7016 & & \\
\overline{A}_{41}^{01} &= 0.196752 & \overline{A}_{41}^{02} &= 0.358682 & \overline{A}_{46}^{01} &= 0.202971 \\
\overline{A}_{46}^{02} &= 0.374801 & \overline{A}_{46}^{12} &= 0.468071 & &
\end{aligned}$$

where 0 is healthy, 1 is critically ill, and 2 is dead. Calculate the premium for a 5-year policy for a life aged 41, with continuous premiums payable while in the healthy state, which pays a benefit \$280,000 immediately upon death in the case of death directly from the healthy state and a benefit of \$190,000 upon entry to the critically ill state, followed by a further benefit of \$140,000 upon death after diagnosis of critical illness. [Hint: You need to separate the death benefits into two cases — cases where the life is critically ill first, and cases where the life is not critically ill first. You can calculate the value for cases where the life is not critically ill first by calculating the value of a payment upon first exit from State 0, which can be calculated from $a_{41:\overline{5}|}^{00}$.

We first calculate $a_{41:\overline{5}|}^{00} = a_{41}^{00} - 0.866102a_{46}^{00}e^{-0.2} = 13.5501 - 0.866102 \times 13.1355e^{-0.2} = 4.23566$

The EPV of the critical illness benefits is given by calculating $\overline{A}_{41:\overline{5}|}^{01} = \overline{A}_{41}^{01} - {}_5p_{41}^{00}e^{-0.2}\overline{A}_{46}^{01} = 0.196752 - 0.866102 \times 0.202971e^{-0.2} = 0.05282438$. The EPV of the critical illness benefits is $0.05282438 \times 190000 = 10036.63$.

The EPV of death benefits is harder to calculate, since the death benefits are reduced if the individual becomes critically ill before death. We have that $\overline{A}_{41:\overline{5}}^{02} = \overline{A}_{41}^{02} - 0.866102e^{-0.2}\overline{A}_{46}^{02} - 0.0542667e^{-0.2}\overline{A}_{46}^{12} = 0.358682 - 0.866102e^{-0.2}0.374801 - 0.0542667e^{-0.2}0.468071 = 0.1137053$.

We can solve the problem of what proportion of these are critically ill first by considering the value of a payment immediately upon any exit from the healthy state. This is given by $1-\delta a_{41}^{00} = 1-0.04\times 13.5501 = 1-0.542004 = 0.457996$. Subtracting the value of payments for entry to the critically ill state gives the EPV of payments for death directly from the healthy state as $0.457996 - 0.196752 = 0.261244$ and for payments for death directly from the healthy state for individuals aged 46, we get $1 - \delta a_{46}^{00} = 1 - 0.04 \times 13.1355 = 1 - 0.52542 = 0.47458$, and $0.47458 - 0.202971 = 0.271609$. Total payments from direct death are therefore given by $0.261244 - 0.866102e^{-0.2} \times 0.271609 = 0.06864488$

Total payments from deaths that are critically ill first are therefore $0.1137053 - 0.06864488 = 0.04506041$.

The total EPV of death benefits is therefore $0.06864488 \times 280000 + 0.04506041 \times 140000 = 25529.02$.

The total EPV of all benefits is therefore, $25529.02 + 10036.63 = \$35,565.65$, so the premium is $\frac{35565.65}{} = \$8,396.72$.

6. *The following is a multiple decrement table giving probabilities of surrender (decrement 1) and death (decrement 2) for a life insurance policy:*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
49	*10000.00*	*235.54*	*1.46*
50	*9763.00*	*222.44*	*1.55*
51	*9539.01*	*210.28*	*1.65*
52	*9327.08*	*198.99*	*1.77*

A life insurance policy has a death benefit of \$400,000 payable at the end of the year of death. Premiums are payable at the beginning of each year. Calculate the premium for a 4-year annual policy sold to a life aged 49 if there is no-payment to policyholders who surrender their policy, and the interest rate is $i = 0.06$.

The expected death benefit is $A^{02}_{49:\overline{4|}} = 0.000146(1.06)^{-1} + 0.000155(1.06)^{-2} + 0.000165(1.06)^{-3} + 0.000177(1.06)^{-4} = 0.0005544231$, so $400000 \times 0.0005544231 = \$221.77$. $a^{00}_{49:\overline{4|}} = 1 + 0.9763(1.06)^{-1} + 0.953901(1.06)^{-2} + 0.932708(1.06)^{-3} = 3.553126$, so the annual premium is $\frac{221.77}{3.553126} = \$62.42$.

7. *Update the multiple decrement table below*

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
58	*10000.00*	*176.04*	*2.68*
59	*9823.96*	*167.67*	*2.88*
60	*9656.29*	*159.84*	*3.10*
61	*9496.46*	*152.50*	*3.34*
62	*9343.96*	*145.62*	*3.60*
63	*9198.34*	*139.16*	*3.89*

with the following mortality probabilities

x	l_x	d_x
58	*10000.00*	*1.81*
59	*9998.19*	*1.92*
60	*9996.27*	*2.04*
61	*9994.22*	*2.18*
62	*9992.05*	*2.32*
63	*9989.73*	*2.47*

[The first decrement is surrender, the second is death.] Using:

(a) UDD in the multiple decrement table.

Under UDD in the multiple decrement table, the relation between single decrements and multiple decrements is.

$$1 - q^{1*} = (1 - q^{01} - q^{02})^{\frac{q^{01}}{q^{01}+q^{02}}}$$

$$1 - q^{2*} = (1 - q^{01} - q^{02})^{\frac{q^{02}}{q^{01}+q^{02}}}$$

We therefore find that the individual decrement probabilities for surrender are

x	q_x^{*1}
58	0.017342321274
59	0.016781677242
60	0.016239569853
61	0.015717162596
62	0.0152112926
63	0.015132032898

Now we update the table with these surrender probabilities and the new death probabilities. The new multiple decrement probabilities are the solution to

$$1 - q^{1*} = (1 - q^{01} - q^{02})^{\frac{q^{01}}{q^{01}+q^{02}}}$$
$$1 - q^{2*} = (1 - q^{01} - q^{02})^{\frac{q^{02}}{q^{01}+q^{02}}}$$

Which are given by

$$1 - q^{01} - q^{02} = (1 - q^{*1})(1 - q^{*2})$$
$$\frac{q^{01}}{q^{02}} = \frac{\log(1 - q^{*1})}{\log(1 - q^{*2})}$$

This gives us

x	p_x^{00}
58	0.982479817686
59	0.983029510665
60	0.983559668135
61	0.98406813965
62	0.984560054641
63	0.984624454627

The new multiple decrement table is therefore

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
10000.00	173.39	1.81	
9824.80	164.85	1.89	
9658.07	156.81	1.97	
9499.28	149.27	2.07	
9347.94	142.16	2.17	
9203.61	139.24	2.28	

(b) UDD in the independent decrements.

For UDD in the individual decrement tables, the decrement probabilities are related by:

$$
\begin{aligned}
q^{01} + q^{02} &= 1 - (1 - q^{*1})(1 - q^{*2}) \\
\frac{q^{01}}{q^{02}} &= \frac{q^{*1}}{q^{*2}} \\
\left(1 - \frac{q^{02}}{q^{01}} q^{*1}\right)(1 - q^{*1}) &= 1 - q^{01} - q^{02} \\
\frac{q^{02}}{q^{01}}(q^{*1})^2 - \left(1 + \frac{q^{02}}{q^{01}}\right) q^{*1} + q^{01} + q^{02} &= 0 \\
q^{*1} &= \frac{q^{01} + q^{02} - \sqrt{(q^{01} + q^{02})^2 - 4q^{01}q^{02}(q^{01} + q^{02})}}{2q^{02}}
\end{aligned}
$$

We therefore find that the individual decrement probabilities for surrender are

x	q_x^{*1}
58	0.01760865
59	0.01707704
60	0.01656770
61	0.01607883
62	0.01561033
63	0.01516079

Now we update the table with these surrender probabilities and the new death probabilities. The new multiple decrement probabilities are given by

$$
\begin{aligned}
q^{01} + q^{02} &= 1 - (1 - q^{*1})(1 - q^{*2}) \\
\frac{q^{01}}{q^{02}} &= \frac{q^{*1}}{q^{*2}} \\
q^{02} &= \frac{1 - (1 - q^{*1})(1 - q^{*2})}{\left(1 + \frac{q^{*1}}{q^{*2}}\right)} \\
&= q^{*2}\left(\frac{q^{*1} + q^{*2} - q^{*1}q^{*2}}{q^{*2} + q^{*1}}\right)
\end{aligned}
$$

The new multiple decrement table is therefore

x	l_x	$d_x^{(1)}$	$d_x^{(2)}$
10000.00	176.05	1.81	
9822.14	167.70	1.89	
9652.55	159.89	1.97	
9490.69	152.57	2.07	
9336.05	145.71	2.17	
9188.18	139.27	2.27	

8. *The mortalities for a husband and wife (whose lives are assumed to be independent) aged 62 and 53 respectively, are given in the following tables:*

x	l_x	d_x
62	*10000.00*	*5.31*
63	*9994.69*	*5.76*
64	*9988.93*	*6.25*
65	*9982.68*	*6.79*
66	*9975.89*	*7.37*
67	*9968.52*	*8.01*

x	l_x	d_x
53	*10000.00*	*3.03*
54	*9996.97*	*3.25*
55	*9993.72*	*3.48*
56	*9990.24*	*3.74*
57	*9986.49*	*4.03*
58	*9982.47*	*4.33*

The interest rate is $i = 0.03$.

(a) They want to purchase a 5-year joint life insurance policy with a death benefit of \$2,500,000. Annual premiums are payable while both are alive. Calculate the net premium for this policy using the equivalence principle.

The EPV of the premium is given by $\ddot{a}_{62,53:\overline{5}|} = 1+0.999469\times 0.999697(1.03)^{-1}+0.998893\times 0.999372(1.03)^{-2}+0.998268\times 0.999024(1.03)^{-3}+0.997589\times 0.998649(1.03)^{-4} = 4.708838$

We also calculate

$A_{62,53:\overline{5}|} = (1-0.999469\times 0.999697)(1.03)^{-1}+(0.999469\times 0.999697-0.998893\times 0.999372)(1.03)^{-2}+(0.998893\times 0.999372-0.998268\times 0.999024)(1.03)^{-3}+(0.998268\times 0.999024-0.997589\times 0.998649)(1.03)^{-4}+(0.997589\times 0.998649-0.996852\times 0.998247)(1.03)^{-5}+ = 0.004463484$

So the EPV of the benefits is $2500000\times 0.004463484 = \$11,158.71$, so the premium is

$$\frac{11158.71}{4.708838} = \$2,369.74$$

(b) They want to purchase a 5-year reversionary annuity, which will provide an annuity to the husband of \$60,000 at the end of each year for the 5-year term if the wife is dead and the husband is alive. Calculate the net annual premium for this policy using the equivalence principle.

The probability that the wife is dead and the husband is alive at the end of each year is given in the following table:

Year	Probability	EPV of payment
1	$(1-0.999697)\times 0.999469 = 0.000303$	$0.000303(1.03)^{-1} = 0.000294$
2	$(1-0.999372)\times 0.998893 = 0.000627$	$0.000627(1.03)^{-2} = 0.000591$
3	$(1-0.999024)\times 0.998268 = 0.000974$	$0.000974(1.03)^{-3} = 0.000892$
4	$(1-0.998650)\times 0.997589 = 0.001347$	$0.001347(1.03)^{-4} = 0.001197$
5	$(1-0.998247)\times 0.996852 = 0.001747$	$0.001747(1.03)^{-5} = 0.001507$

Summing the last column gives $a^{01}_{63,52:\overline{5}|} = 0.004480903$, so the EPV of benefits is $0.004480903\times 60000 = \$268.85$, and the premium is $\frac{268.85}{4.708838} = \$57.10$.

(c) They want to purchase a 5-year last survivor insurance policy, with a death benefit of \$120,000,000. Premiums are payable while either life is alive. Calculate the net premium for this policy using the equivalence principle.

Year	Probability Both Dead	Probability of Payment	EPV of payment	EPV of premium
1	$(1-0.999469)\times(1-0.999697) = 0.000000161$	0.000000161	0.000000156	0.9708736
2	$(1-0.998893)\times(1-0.999372) = 0.000000695$	0.000000534	0.000000504	0.9425953
3	$(1-0.998268)\times(1-0.999024) = 0.000001690$	0.000000995	0.000000911	0.9151401
4	$(1-0.997589)\times(1-0.998649) = 0.000003257$	0.000001567	0.000001390	0.8884842
5	$(1-0.996852)\times(1-0.998247) = 0.000005518$	0.000002261	0.000001953	0.8626040

This gives $\ddot{a}_{\overline{62,53:\overline{5}}} = 4.717093$ and $A_{\overline{62,53:\overline{5}}} = 0.000004913182$, so the premium is $\frac{0.000004913182\times 120000000}{4.717093} =$ \$124.99.

9. *A husband is 64; the wife is 73. Their lifetables while both are alive, and the lifetable for the husband if the wife is dead, are given below:*

x	l_x	d_x
64	*10000.00*	*6.92*
65	*9993.08*	*7.49*
66	*9985.59*	*8.12*
67	*9977.48*	*8.80*
68	*9968.68*	*9.55*
69	*9959.13*	*10.36*

x	l_x	d_x
73	*10000.00*	*31.73*
74	*9968.27*	*34.69*
75	*9933.58*	*37.92*
76	*9895.66*	*41.45*
77	*9854.20*	*45.30*
78	*9808.91*	*49.49*

x	l_x	d_x
64	*10000.00*	*11.56*
65	*9988.44*	*12.56*
66	*9975.88*	*13.65*
67	*9962.23*	*14.83*
68	*9947.40*	*16.12*
69	*9931.28*	*17.53*

Calculate the probability that the husband survives to the end of the 5-year period. Use the UDD assumption for handling changes to the husband's mortality in the event of the wife's death.

In the year of the wife's death, let q_a be the probability that the husband dies if the wife is alive for the whole year, and let q_d be the probability that the husband dies if the wife is dead for the whole year. If the wife's death occurs at time t, then the husband's probability of surviving the year is $(1-tq_a)\frac{1-q_d}{1-tq_d}$. The overall probability of the husband surviving the year if the wife's death is uniformly distributed is therefore

$$\begin{aligned}\int_0^1 (1-q_d)\frac{1-tq_a}{1-tq_d}\,dt &= (1-q_d)\int_0^1 \left(\frac{q_a}{q_d} - \left(\frac{q_a-q_d}{{q_d}^2}\right)\frac{q_d}{1-tq_d}\right)dt\\ &= (1-q_d)\left(\frac{q_a}{q_d} - \left(\frac{q_a-q_d}{{q_d}^2}\right)\log(1-tq_d)\right)\end{aligned}$$

This gives the following table:

Year	P(W Dies)	P(H survives to start of year)	P(H survives year)	P(H survives from end of year)	Total P(H survives 5 years)
1	0.003173	1.000000	0.9990759	$\frac{9931.28}{9988.44}$	0.003151927
2	0.003469	0.999308	0.9989964	$\frac{9931.28}{9975.88}$	0.003447638
3	0.003792	0.998559	0.9989091	$\frac{9931.28}{9962.23}$	0.003770620
4	0.004145	0.997748	0.9988145	$\frac{9931.28}{9947.40}$	0.004124069
5	0.004530	0.996868	0.9987106	$\frac{9931.28}{9931.28}$	0.004509989
>5	0.980891	0.995913	1	1	0.976882098

The total probability that the husband survives the 5 years is therefore $0.003151927 + 0.003447638 + 0.003770620 + 0.004124069 + 0.004509989 + 0.976882098 = 0.9958863$.

10. *A couple want to receive the following:*

- *While both are alive, they would like to receive a pension of \$90,000 per year.*
- *If the husband is alive and the wife is not, they would like to receive a pension of \$85,000 per year.*
- *If the wife is alive and the husband is not, they would like to receive a pension of \$65,000 per year.*

- *When one dies, if the husband dies first, they would like to receive \$92,000, if the wife dies first, they would like to receive \$120,000.*
- *When the second one dies, if it is the husband, they would like to receive a benefit of \$65,000; if it is the wife, they would like to receive a benefit of \$93,000.*

Construct a combination of insurance and annuity policies that achieve this combination of benefits.

There are many possible solutions. Below are two of them.

First solution:

- A last survivor annuity for \$65,000.
- A life annuity for the husband for \$20,000.
- A joint life annuity for \$5,000.
- A life insurance policy for \$65,000 for the husband.
- A life insurance policy for \$93,000 for the wife.
- A joint life insurance policy for \$27,000.

This gives the following:

- While both are alive, an annuity of $5000 + 65000 + 20000 = 90,000$.
- While the wife is dead and the husband is alive, an annuity of $65000 + 20000 = 85,000$.
- While the wife is alive and the husband is dead, an annuity of $\$65,000$.
- If the wife dies first, a death benefit of $93000 + 27000 = \$120,000$.
- If the husband dies first, a death benefit of $65000 + 27000 = \$92,000$.
- If the wife dies second, a death benefit of $\$93,000$.
- If the husband dies second, a death benefit of $\$65,000$.

Second solution:

- A life annuity for the wife for \$5,000.
- A life annuity for the husband for \$85,000.
- A reversionary annuity for \$60,000 while the husband is dead and the wife is alive.
- A last survivor insurance policy for \$65,000.
- A life insurance policy for \$28,000 for the wife.
- A joint life insurance policy for \$92,000.

This gives the following:

- While both are alive, an annuity of $85000 + 5000 = \$90,000$.
- While the wife is dead and the husband is alive, an annuity of $\$85,000$.
- While the wife is alive and the husband is dead, an annuity of $\$65,000$.
- If the wife dies first, a death benefit of $92000 + 28000 = \$120,000$.
- If the husband dies first, a death benefit of $\$92,000$.
- If the wife dies second, a death benefit of $65000 + 28000 = \$93,000$.

- If the husband dies second, a death benefit of \$65,000.

11. *A husband aged 52 and wife aged 66 have the following transition intensities:*

$$\begin{aligned}
\mu_{xy}^{01} &= 0.000003y + 0.0000001x \\
\mu_{xy}^{02} &= 0.0000015x + 0.0000004y \\
\mu_{xy}^{03} &= 0.000042 + 0.000013x + 0.000019y \\
\mu_{x}^{13} &= 0.000004x \\
\mu_{x}^{23} &= 0.000003y
\end{aligned}$$

Which of the following expressions gives the probability that after 7 years, the husband is dead and the wife is alive? Justify your answer.
(i) $\int_0^7 e^{-(0.0014595+0.0020203t+0.0000205t^2)}(0.00001044+0.0000039t)\,dt$
(ii) $\int_0^7 e^{-(0.0023614+0.0014475t+0.0000205t^2)}(0.00001044+0.0000019t)\,dt$
(iii) $\int_0^7 e^{-(0.0014595+0.0020996t+0.0000170t^2)}(0.00001044+0.0000019t)\,dt$
(iv) $\int_0^7 e^{-(0.0009948+0.0020203t+0.0000150t^2)}(0.00001044+0.0000019t)\,dt$
The probability of this is

$$\begin{aligned}
&\int_0^7 {}_tp_{52,66}^{00}\mu_{52+t,66+t}^{02}\,{}_{7-t}p_{52+t,66+t}^{22}\,dt \\
=&\int_0^7 e^{-\int_0^t 0.0000224(66+s)+0.0000146(52+s)+0.000042\,ds}(0.0000015(52+t)+0.0000004(66+t))e^{-\int_t^7 0.000003(66+s)\,ds}\,d \\
=&\int_0^7 e^{-\int_0^t 0.0022796+0.0000370s\,ds}(0.0001044+0.0000019t)e^{-\int_t^7 0.000198+0.000003s\,ds}\,dt \\
=&\int_0^7 e^{-(0.0022976t+0.0000185t^2)}(0.0001044+0.0000019t)e^{-(0.000198(7-t)+0.0000015(49-t^2))}\,dt \\
=&\int_0^7 e^{-(0.0014595+0.0020996t+0.0000170t^2)}(0.00001044+0.0000019t)\,dt
\end{aligned}$$

So the answer is (iii).

12. *A life aged 38 wants to buy a 3-year term insurance policy. A life-table based on current-year mortality is:*

x	l_x	d_x
38	10000.00	5.00
39	9995.00	5.14
40	9989.86	5.30
41	9984.56	5.47
42	9979.09	5.67
43	9973.42	5.87

The insurance company uses a single-factor scale function $q(x,t) = q(x,0)(1-\phi_x)^t$ to model changes in mortality. The insurance company uses the following values for ϕ_x:

x	ϕ_x
38	*0.03*
39	*0.025*
40	*0.025*
41	*0.02*
42	*0.015*
43	*0.02*

Calculate $A^{1}_{38:\overline{3}|}$ at interest rate $i = 0.06$, taking into account the change in mortality.

We have $q_{38} = 0.0005$, $q_{39}(1-\phi_{39}) = 0.975 \times \frac{5.14}{9995.00} = 0.00050140070035$, $q_{40}(1-\phi_{40})^2 = 0.975^2 \times \frac{5.30}{9989.86} = 0.000504342653451$. This gives $A^{1}_{38:\overline{3}|} = 0.0005(1.06)^{-1} + (1-0.0005) \times 0.00050140070035(1.06)^{-2} + (1-0.0005)(1-0.00050140070035) \times 0.000504342653451(1.06)^{-2} = 0.00136613361588$.

13. *The following lifetable applied in 2016:*

x	l_x	d_x
55	*10000.00*	*10.63*
56	*9989.37*	*11.30*
57	*9978.07*	*12.02*
58	*9966.05*	*12.80*
59	*9953.25*	*13.66*
60	*9939.59*	*14.60*

An insurance company uses the following mortality scale based on both age and year:

	t					
x	*2017*	*2018*	*2019*	*2020*	*2021*	*2022*
55	0.01	0.015	0.015	0.02	0.02	0.015
56	0.03	0.03	0.025	0.02	0.015	0.02
57	0.02	0.03	0.03	0.025	0.02	0.015
58	0.025	0.03	0.025	0.015	0.015	0.02
59	0.015	0.02	0.015	0.01	0.015	0.01
60	0.02	0.015	0.01	0.015	0.02	0.025

Use this mortality scale to calculate $A^{1}_{55:\overline{4}|}$ at interest rate $i = 0.03$.

For this individual we have

$$q_{55} = (1-0.01)0.001063 = 0.00105237$$
$$q_{56} = (1-0.03)(1-0.03) \times \frac{11.30}{9989.37} = 0.00106434840234$$
$$q_{57} = (1-0.02)(1-0.03)(1-0.03) \times \frac{12.02}{9978.07} = 0.00111077850125$$
$$q_{58} = (1-0.025)(1-0.03)(1-0.025)(1-0.015) \times \frac{12.80}{9966.05} = 0.00116655200405$$

This gives $A^{1}_{55:\overline{4}|} = 0.00105237(1.03)^{-1} + (1-0.00105237)\big(0.00106434840234(1.03)^{-2} + (1-0.00106434840234)\big(0$ 0.00420107324841.

14. A pensions company has the current mortality scale for 2017:

x	$\phi(x, 2017)$	$\left.\frac{d\phi(x,t)}{dt}\right\vert_{x,t=2017}$	$\left.\frac{d\phi(x+t,t)}{dt}\right\vert_{x,t=2017}$
51	*0.016389776*	0.00054272913	−0.0015000971
52	*0.018738397*	−0.00107674028	0.0012410504
53	*0.028229446*	0.00120650853	−0.0002976607
54	*0.028011768*	−0.00109930339	−0.0004183465
55	*0.014334489*	−0.00194027424	0.0023952205
56	*0.016770205*	0.00271342277	−0.0053102487

Mortality in 2016 is given in the following lifetable.

x	l_x	d_x
51	*10000.00*	*15.29*
52	*9984.71*	*16.44*
53	*9968.27*	*17.70*
54	*9950.56*	*19.09*
55	*9931.48*	*20.60*
56	*9910.88*	*22.26*

The company assumes that from 2030 onwards, we will have $\phi(x,t) = 0.01$ *for all* x *and* t. *Calculate* $q(54, 2018)$ *using the average of age-based and cohort-based effects.*

We fit a cubic curve between the known points. For age 54, we have $\phi(54, 2017+t) = f(t) = at^3 + bt^2 + ct + d$, and we get

$$\begin{aligned} f(0) &= 0.028011768 \\ f'(0) &= -0.00109930339 \\ f(13) &= 0.01 \\ f'(13) &= 0 \end{aligned}$$

We solve this to get

$$\begin{aligned} d &= 0.028011768 \\ c &= -0.00109930339 \\ 13^3a + 13^2b + 13c + d &= 0.01 \\ 3 \times 13^2a + 2 \times 13b + c &= 0 \\ 13^3a - 13c - 2d &= -0.02a \qquad = \frac{13 \times -0.}{} \\ b &= \frac{0.00109930339 - 3 \times 13^2 \times 0.00000989193988621}{2 \times 13} = -0.000150611928166 \end{aligned}$$

This gives $f(1) = 0.00000989193988621 - 0.000150611928166 - 0.00109930339 + 0.028011768 = 0.0267717446217$.

For the cohort-based curve, we have $\phi(53+t, 2017+t) = g(t) = \tilde{a}t^3 + \tilde{b}t^2 + \tilde{c}t + \tilde{d}$ and we get

$$\begin{aligned} g(0) &= 0.028229446 \\ g'(0) &= -0.0002976607 \\ g(13) &= 0.01 \\ g'(13) &= 0 \end{aligned}$$

We solve this to get

$$\begin{aligned} \tilde{d} &= 0.028229446 \\ \tilde{c} &= -0.0002976607 \\ 13^3\tilde{a} + 13^2\tilde{b} + 13\tilde{c} + \tilde{d} &= 0.01 \\ 3 \times 13^2\tilde{a} + 2 \times 13\tilde{b} + \tilde{c} &= 0 \\ 13^3\tilde{a} - 13\tilde{c} - 2\tilde{d} &= -0.02\tilde{a} \qquad = \frac{13 \times -0.000}{} \\ \tilde{b} &= \frac{0.0002976607 - 3 \times 13^2 \times 0.0000148335470642}{2 \times 13} = -0.00027780567929 \end{aligned}$$

This gives $g(1) = 0.0000148335470642 - 0.00027780567929 - 0.0002976607 + 0.028229446 = 0.0276688131678$. Taking the average of the age-based and cohort-based improvement factors, we get $\phi(54, 2018) = \frac{0.0267717446217+0.0276688131678}{2} = 0.0272202788948$. We therefore have

$$q(54, 2018) = q(54, 2016)(1-\phi(54, 2017))(1-\phi(54, 2018)) = \frac{19.09}{9950.56}(1-0.028011768)(1-0.0272202788948) = 0.00$$

15. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.6 \qquad \sigma_k = 1.4 \qquad K_{2017} = -4.83$$

And the following values of α_x and β_x:

x	α_x	β_x
34	−5.314675	0.2697754
35	−5.234098	0.2504377
36	−5.043921	0.1782635
37	−4.892803	0.2889967
38	−4.637988	0.1460634
39	−4.413315	0.1174245
40	−4.261060	0.2078267

The insurance company simulates the following values of Z_t:

t	Z_t
2018	0.2525295
2019	−0.6276655
2020	−0.6007807

Using these simulated values, calculate the probability that a life aged exactly 36 at the start of 2017 dies within the next 4 years.

From the simulated values we have

$$\begin{aligned}
K_{2018} &= -4.83 - 0.6 + 1.4 \times 0.2525295 = -5.0764587 \\
K_{2019} &= -5.0764587 - 0.6 + 1.4 \times -0.6276655 = -6.5551904 \\
K_{2020} &= -6.5551904 - 0.6 + 1.4 \times -0.6007807 = -7.99628338
\end{aligned}$$

This gives us

$$\begin{aligned}
\log(m(36, 2017)) &= -5.043921 + 0.1782635 \times -4.83 = -5.904933705 \\
\log(m(37, 2018)) &= -4.892803 + 0.2889967 \times -5.0764587 = -6.35988281199 \\
\log(m(38, 2019)) &= -4.637988 + 0.1460634 \times -6.5551904 = -5.59546139747 \\
\log(m(39, 2020)) &= -4.413315 + 0.1174245 \times -7.99628338 = -5.35227457776
\end{aligned}$$

Under UDD, we have $m_x = \frac{2q_x}{2-q_x}$ so $q_x = \frac{2m_x}{2+m_x}$. This gives us

$$\begin{aligned}
q(36, 2017) &= \frac{2e^{-5.904933705}}{2 + e^{-5.904933705}} = 0.00272225211385 \\
q(37, 2018) &= \frac{2e^{-6.35988281199}}{2 + e^{-6.35988281199}} = 0.001728074972 \\
q(38, 2019) &= \frac{2e^{-5.59546139747}}{2 + e^{-5.59546139747}} = 0.00370779834235 \\
q(39, 2020) &= \frac{2e^{-5.35227457776}}{2 + e^{-5.35227457776}} = 0.00472616844589
\end{aligned}$$

The probability that the life survives four years is therefore

$$(1-0.00272225211385)(1-0.001728074972)(1-0.00370779834235)(1-0.00472616844589) = 0.987175350394$$

16. *An insurance company uses a Lee-Carter model. One actuary fits the following parameters:*

$$c = -0.13 \qquad \sigma_k = 0.9 \qquad K_{2017} = -1.70 \qquad \alpha_{52} = -4.45 \qquad \beta_{52} = 0.49$$

A second actuary fits the parameters

$$c = -0.14 \qquad \sigma_k = 0.8 \qquad K_{2017} = -1.40 \qquad \alpha_{52} = -4.94 \qquad \beta_{52} = 0.37$$

The insurance company sets its life insurance premiums for 2025 so that under the first actuary's model, it has a 95% chance of an expected profit. What is the probability that these premiums lead to an expected profit under the second actuary's model?

Since expected profit is a decreasing function of $m(x,t)$, we need to calculate the probability under the second actuary's model that $\log(m(52,2025))$ is less than the 95th percentile of the first actuary's distribution for $\log(m(52,2025))$.

The first actuary's model gives $\log(m(52,2025)) = -4.45+0.49K_{2025}$, where $K_{2025} = -1.70-0.13\times 8+0.9(Z_{2018}+Z_{2019}+Z_{2020}+Z_{2021}+Z_{2022}+Z_{2023}+Z_{2024}+Z_{2025}) \sim N(-2.74, 6.48)$. The 95th percentile of this model is therefore $\log(m(52,2025)) = -4.45+0.49(-2.74+1.96\sqrt{6.48}) = -3.34782073046$. The second actuary's model gives $\log(m(52,2025)) = -4.94+0.37K_{2025}$, where $K_{2025} = -1.40-0.14\times 8+0.8(Z_{2018}+Z_{2019}+Z_{2020}+Z_{2021}+Z_{2022}+Z_{2023}+Z_{2024}+Z_{2025}) \sim N(-2.52, 5.12)$. Under this model $\log(m(52,2025)) \sim N(-5.8724, 5.12\times 0.37^2)$. The probability that $\log(m(52,2025)) < -3.34782073046$ is therefore

$$\Phi\left(\frac{-3.34782073046-(-5.8724)}{0.37\sqrt{5.12}}\right) = \Phi(3.01545121813) = 0.998717$$

17. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$K^{(1)}_{2017} = -3.29$	$K^{(2)}_{2017} = 0.38$	$c^{(1)} = -0.17$	$c^{(2)} = 0.01$
$\sigma_{k_1} = 0.5$	$\sigma_{k_2} = 0.08$	$\rho = 0.3$	$\overline{x} = 47$

What is the probability that the mortality for an individual currently (in 2017) aged 39 will be higher in 2025 than in 2030?

The mortality in 2025 satisfies

$$\log\left(\frac{q(47,2025)}{1-q(47,2025)}\right) = K^{(1)}_{2025}$$

while mortality in 2030 satisfies

$$\log\left(\frac{q(52,2030)}{1-q(52,2030)}\right) = K^{(1)}_{2030} + 5K^{(2)}_{2030}$$

Since $\log\left(\frac{q}{1-q}\right)$ is an increasing function of q, we are asking, what is the probability that $K^{(1)}_{2025} > K^{(1)}_{2030}+5K^{(2)}_{2030}$. We have that $K^{(1)}_{2030} = K^{(1)}_{2025}-5\times 0.17+0.5(Z^{(1)}_{2026}+Z^{(1)}_{2027}+Z^{(1)}_{2028}+Z^{(1)}_{2029}+Z^{(1)}_{2030})$ and $K^{(2)}_{2030} = K^{(2)}_{2017}+13\times 0.01+0.08(Z^{(2)}_{2018}+\cdots+Z^{(2)}_{2030})$. The probability that we are interested in is therefore the probability that

$$-0.85+0.5(Z^{(1)}_{2026}+Z^{(1)}_{2027}+Z^{(1)}_{2028}+Z^{(1)}_{2029}+Z^{(1)}_{2030})+5\left(0.51+0.08(Z^{(2)}_{2018}+\cdots+Z^{(2)}_{2030})\right) < 0$$

$$1.7+0.5(Z^{(1)}_{2026}+Z^{(1)}_{2027}+Z^{(1)}_{2028}+Z^{(1)}_{2029}+Z^{(1)}_{2030})+0.4(Z^{(2)}_{2018}+\cdots+Z^{(2)}_{2030}) < 0$$

We have that $\mathrm{Cov}(Z^{(1)}_t, Z^{(2)}_t) = 0.3$, so $\mathrm{Var}(0.5Z^{(1)}_t+0.4Z^{(2)}_t) = 0.5^2+0.4^2+2\times 0.3\times 0.4\times 0.5 = 0.53$. The variance of $0.5(Z^{(1)}_{2026}+Z^{(1)}_{2027}+Z^{(1)}_{2028}+Z^{(1)}_{2029}+Z^{(1)}_{2030})+0.4(Z^{(2)}_{2018}+\cdots+Z^{(2)}_{2030})$ is therefore $8\times 0.4^2+5\times 0.5 = 3.93$ We therefore get that

$$\log\left(\frac{q(52,2030)}{1-q(52,2030)}\right) - \log\left(\frac{q(47,2025)}{1-q(47,2025)}\right)) \sim N(1.7, 3.93)$$

The probability that it is less than 0 is therefore $\Phi\left(\frac{-1.7}{\sqrt{3.93}}\right) = \Phi(-0.857536562919) = 0.1955742$.

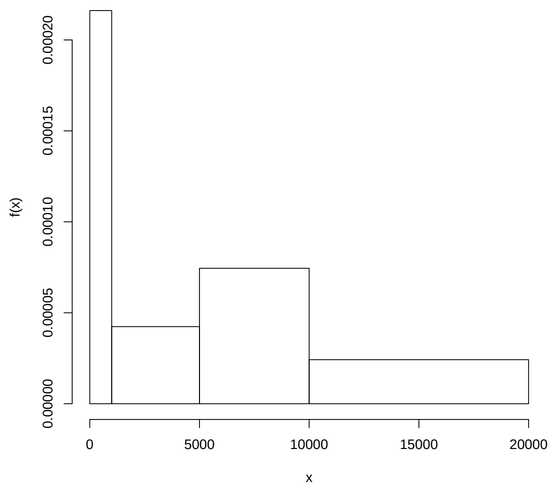
18. *For the following dataset:*

    0.2 0.2 0.4 0.7 1.8 2.1 2.3 3.0 3.5 3.9 4.1 4.2 4.6 5.1 5.7 6.6 8.2 11.4

    Calculate a Nelson-Åalen estimate for the probability that a random sample is more than 2.7.

    The Nelson-Åalen estimate for $H(2.7)$ is $\frac{2}{18}+\frac{1}{16}+\frac{1}{15}+\frac{1}{14}+\frac{1}{13}+\frac{1}{12}=0.4719628$, so the survival functions is $S(2.7)=e^{-0.4719628}=0.6237767$.

19. *The histogram below is obtained from a sample of 8,000 claims.*



    Which interval included most claims?

    The probability of the first interval is approximately $0.00022 \times 1000 = 0.22$.

    The probability of the second interval is approximately $0.00004 \times 4000 = 0.16$.

    The probability of the third interval is approximately $0.00007 \times 5000 = 0.35$.

    The probability of the first interval is approximately $0.00003 \times 10000 = 0.30$.

    Therefore the interval 5000–10000 included most claims.

20. *An insurance company collects the following data on insurance claims:*

Claim Amount	*Number of Policies*
Less than \$5,000	*232*
\$5,000–\$20,000	*147*
\$20,000–\$100,000	*98*
More than \$100,000	*23*

    The policy currently has no deductible and a policy limit of \$100,000. The company wants to determine how much would be saved by introducing a deductible of \$2,000 and a policy limit of \$50,000. Using the ogive to estimate the empirical distribution, how much would the expected claim amount be reduced by the new deductible and policy limit?

    Using the ogive, the current expected claim amount is $\frac{232\times2500+147\times12500+98\times60000+23\times100000}{500}=$ $\$21,195.00$.

    Using the ogive, the expected number of claims between 0 and 2000 is $\frac{2000}{5000}\times 232 = 92.8$. The number of claims above 52000 (50000 policy limit plus 2000 deductible) is $\frac{52000}{80000}\times 98+23=$

86.7. The expected claim amount per loss is therefore $\frac{139.2\times1500+147\times10500+24.3\times33000+86.7\times50000}{500} =$ \$13,778.40, so the reduction is $21195.00 - 13778.40 = \$7,416.6$.

21. *An insurance company collects the following claim data (in thousands):*

i	d_i	x_i	u_i	i	d_i	x_i	u_i	i	d_i	x_i	u_i
1	0	0.8	-	8	0.5	-	5	15	2.0	-	5
2	0	1.3	-	9	1.0	1.2	-	16	2.0	-	10
3	0	-	20	10	1.0	-	15	17	2.0	2.4	-
4	0	4.4	-	11	1.0	1.8	-	18	2.0	-	5
5	0	-	10	12	1.0	-	10	19	2.0	11.6	-
6	0.5	1.4	-	13	1.0	6.3	-	20	5.0	-	15
7	0.5	1.8	-	14	2.0	4.9.	-	21	5.0	5.9	-

Using a Kaplan-Meier product-limit estimator:

(a) estimate the probability that a random loss exceeds 3.

y_i	s_i	r_i
0.8	1	8
1.2	1	12
1.3	1	11
1.4	1	10
1.8	2	9
2.4	1	13

So the Kaplan-Meier estimator is $S(3) = \frac{7}{8} \times \frac{11}{12} \times \frac{10}{11} \times \frac{9}{10} \times \frac{7}{9} \times \frac{12}{13} = \frac{49}{104} = 0.4711538$.

(b) Use Greenwood's approximation to obtain a 95% confidence interval for the probability that a random loss exceeds 3, based on the Kaplan-Meier estimator, using a normal approximation.

Greenwood's approximation gives $\mathrm{Var}(\hat{S}(3)) = (\hat{S}(3))^2 \sum_{i=1}^{6} \frac{s_i}{r_i(r_i-s_i)} = 0.4711538^2 \left(\frac{1}{8\times7} + \frac{1}{12\times11} + \frac{1}{11\times10} + \frac{1}{10\times9}\right.$ 0.01860047.

Using a normal approximation, the confidence interval is $0.4711538 \pm 1.96\sqrt{0.01860047} = [0.2038421, 0.7384655]$.

(c) Use Greenwood's approximation to find a log-transformed confidence interval for the probability that a random loss exceeds 3.

The log-transformed inteval is $[S_n(x)^{\frac{1}{U}}, S_n(x)^{\frac{1}{U}}]$, where $U = e^{1.96\left(\frac{\sqrt{0.01860047}}{S_n(x)\log(S_n(x))}\right)}$. That is

$$\begin{aligned} U &= e^{1.96\left(\frac{\sqrt{0.01860047}}{0.4711538\log(0.4711538)}\right)} \\ &= 0.4705326 \end{aligned}$$

so the confidence interval is

$$[0.4711538^{\frac{1}{0.4705326}}, 0.4711538^{0.4705326}] = [0.2020174, 0.7017985]$$

22. *An insurance company records the following data in a mortality study:*

entry	*death*	*exit*	*entry*	*death*	*exit*	*entry*	*death*	*exit*
51.3	-	*58.4*	*56.5*	-	*58.2*	*55.3*	-	*59.9*
54.7	-	*59.7*	*54.7*	-	*59.8*	*53.3*	*59.1*	
53.8	-	*58.5*	*57.9*	-	*61.3*	*56.7*	*58.4*	-
57.3	-	*58.3*	*58.0*	-	*59.3*	*52.4*	*58.9*	-
52.8	-	*60.6*	*58.4*	-	*59.8*	*57.7*	*58.8*	-
58.7	-	*59.5*	*53.0*	-	*58.3*	*58.3*	*60.4*	-
53.3	-	*62.4*	*53.1*	-	*60.1*	*58.1*	*58.4*	-

Estimate the probability of an individual currently aged exactly 58 dying within the next year using:

(a) the exact exposure method.

The exact exposure is $0.4+1+0.5+0.3+1+0.3+1+0.2+1+1+1+0.6+0.3+1+1+1+0.4+0.9+0.8+0.7+0.3 = 14.7$, and there are 4 deaths at age 58, so the hazard rate is $\frac{4}{14.7}$, and the probability of dying is therefore $1-e^{-\frac{4}{14.7}} = 0.2382287$.

(b) the actuarial exposure method.

The actuarial exposure is $0.4+1+0.5+0.3+1+0.3+1+0.2+1+1+1+0.6+0.3+1+1+1+1+1+1+0.7+0.9 = 16.2$, so the probability of dying is $\frac{4}{16.2} = 0.2469136$.

23. *Using the following table:*

Age	*No. at start*	*enter*	*die*	*leave*	*No. at next age*
48	*26*	*43*	*2*	*13*	*54*
49	*54*	*39*	*7*	*17*	*69*
50	*69*	*46*	*14*	*28*	*73*
51	*73*	*22*	*13*	*44*	*38*

Estimate the probability that an individual aged 49 withdraws from the policy within the next two years, conditional on surviving to the end of those two years.

Using exact exposure

For age 49, the exact exposure is $54+\frac{39-7-17}{2} = 61.5$, and the number of withdrawls is 17, so the hazard rate is $\frac{17}{61.5}$. For age 50, the exact exposure is $69+\frac{46-14-28}{2} = 71$, and the number of withdrawls is 28, so the hazard rate is $\frac{28}{71}$. The probability of not withdrawing in the next two years is therefore $e^{-\frac{17}{61.5}-\frac{28}{71}} = 0.511305$, so the probability of withdrawing during the next two years is $1-0.511305 = 0.488695$.

Using actuarial exposure

For age 49, the actuarial exposure is $54+\frac{39-7}{2} = 70$, and the number of withdrawls is 17, so the probability of withdrawing is $\frac{17}{70}$. For age 50, the actuarial exposure is $69+\frac{46-14}{2} = 85$, and the number of withdrawls is 28, so the probability of withdrawl is $\frac{28}{85}$. The probability of not withdrawing in the next two years is therefore $\frac{53}{70}\times 5785 = 0.5164103$, so the probability of withdrawing during the next two years is $1-0.5164103 = 0.4835897$.