ACSC/STAT 4720, Life Contingencies II

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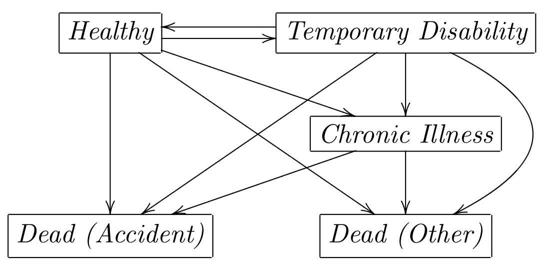
Homework Sheet 1

Model Solutions

Basic Questions

1. *An insurance company provides temporary disability insurance, chronic illness insurance and life insurance. It also pays an additional benefit for accidental death. It has the following model of the states:*



Which of the following sequences of transitions are possible? (Indicate which parts of the transition sequence are not possible if the sequence is not possible.)

(i) Healthy–Temporary Disability – Dead (Accident)

This is possible.

(ii) Healthy–Temporary Disability – Healthy – Chronic Illness

This is possible.

(iii) Temporary Disability – Healthy – Dead (Other) – Chronic Illness

This is impossible, because the transition Dead (Other) – Chronic Illness is not possible.

(iv) Temporary Disability – Healthy – Temporary Disability – Dead (Other)

This is possible.

(v) Healthy–Chronic Illness–Temporary Disability –Healthy

This is impossible, because the transition Chronic Illness–Temporary Disability is not possible.

2. *Consider a critical illness model with transition intensities*

$$\begin{aligned}\mu_x^{01} &= 0.001 + 0.000001x\\ \mu_x^{02} &= 0.002 + 0.000006x\\ \mu_x^{12} &= 0.08 + 0.0002x\end{aligned}$$

where State 0 is healthy, State 1 is critically ill and State 2 is dead.

(a) Calculate the probability that a healthy individual aged 43 is still healthy at age 61.

This is given by

$$e^{-\int_{43}^{61} 0.003+0.000007x\,dx} = e^{-\left[0.003x+0.0000035x^2\right]_{43}^{61}} = e^{-\left(0.003\times 18+0.0000035(61^2-43^2)\right)}$$
$$= e^{-(0.054+0.006552)} = 0.941244823$$

(b) Calculate the probability that a healthy individual aged 38 is dead by age 64.

There are two ways the individual can be dead — dying directly from the healthy state, and becoming critically ill first. If the life becomes critically ill at age a, the probability that the life is still alive at age 64 is given by

$$e^{-\int_a^{64} 0.08+0.0002x\,dx} = e^{-\left(0.08(64-a)+0.0001(64^2-a^2)\right)}$$

The probability density of becoming permanently disabled at age a is

$$(0.001+0.000001a)e^{-\int_{38}^{a} 0.003+0.000007x\,dx} = (0.001+0.000001a)e^{-\left(0.003(a-38)+0.0000035(a^2-38^2)\right)}$$

This means the probability that the life is critically ill at age 64 is

$$\int_{38}^{64}(0.001+0.000001a)e^{-\left(0.003(a-38)+0.0000035(a^2-38^2)\right)}e^{-\left(0.08(64-a)+0.0001(64^2-a^2)\right)}\,da$$
$$=\int_{38}^{64}(0.001+0.000001a)e^{0.114-5.12+0.08a-0.003a+0.005054-0.4096-0.0000035a^2+0.0001a^2}\,da$$
$$=\int_{38}^{64}(0.001+0.000001a)e^{-5.410546+0.077a+0.0000965a^2}\,da$$
$$=\int_{38}^{64}0.000001(a+1000)e^{0.0000965(a+398.96373057)^2-20.7706496269}\,da$$
$$=0.00987437$$

The probability that the life is healthy is given by

$$e^{-\int_{38}^{64} 0.002+0.000007x\,dx} = e^{-\left[0.003x+0.0000035x^2\right]_{38}^{64}} = e^{-\left(0.003\times 26+0.0000035(64^2-38^2)\right)} = e^{-(0.078+0.009282)} = 0.9164186291$$

The probability that the life is dead is therefore $1 - 0.9164186291 - 0.01865412559 = 0.06492724531$.

3. *Under a disability income model with transition intensities*

$$\begin{aligned}
\mu_x^{01} &= 0.001\\
\mu_x^{10} &= 0.016\\
\mu_x^{02} &= 0.003\\
\mu_x^{12} &= 0.009
\end{aligned}$$

calculate the probability that a healthy individual is healthy 7 years later and has not had more than one period of disability during the 7 years. [State 0 is healthy, State 1 is sick and State 2 is dead.]

The probability that an individual remains healthy throughout the 7 years is $e^{-0.004\times 7} = e^{-0.028} = 0.9723883668$. The probability density that the individual becomes disabled at time s, recovers at time t, and remains healthy to the end of the 7-year period is

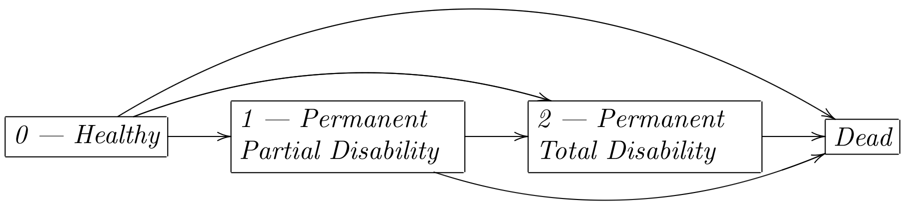
$$0.001 \times 0.016e^{-0.004s}e^{-0.025(t-s)}e^{-0.004(7-t)} = 0.000016e^{-0.028-0.021(t-s)}$$

. Integrating this over s and t gives that the probability that the individual has a single period of disability and is healthy after 7 years is

$$\begin{aligned}
\int_0^7\int_s^7 0.000016e^{-0.028-0.021(t-s)}\,dt\,ds &= \int_0^7\int_s^7 0.000016e^{-0.028+0.021s-0.021t}\,dt\,ds\\
&= 0.000016e^{-0.028}\int_0^7 e^{0.021s}\int_s^7 e^{-0.021t}\,dt\,ds\\
&= 0.000016e^{-0.028}\int_0^7 e^{0.021s}\left[-\frac{e^{-0.021t}}{0.021}\right]_s^7 ds\\
&= 0.000016e^{-0.028}\int_0^7 e^{0.021s}\frac{e^{-0.021s}-e^{-0.147}}{0.021}\,ds\\
&= \frac{0.000016e^{-0.028}}{0.021}\int_0^7 1-e^{-0.147}e^{0.021s}\,ds\\
&= \frac{0.000016e^{-0.028}}{0.021}\left(7-e^{-0.147}\frac{e^{0.147}-1}{0.021}\right)\,ds\\
&= 0.0003631653111
\end{aligned}$$

Thus, the total probability that the individual is healthy after 7 years and has had at most one period of disability is $0.9723883668+0.0003631653111 = 0.972751532111$.

4. *A permanent disability model has the following state diagram:*



The transition intensities at age x are given by:

$$\begin{aligned}
\mu_x^{01} &= 0.001\\
\mu_x^{02} &= 0.002\\
\mu_x^{03} &= 0.003 + 0.000006x\\
\mu_x^{12} &= 0.07\\
\mu_x^{13} &= 0.06\\
\mu_x^{23} &= 0.14 + 0.005x
\end{aligned}$$

calculate the premium for a whole life policy sold to a life aged 42 with premiums payable continuously while the life is in the healthy state, which pays a benefit of \$110,000 upon entry into State 1 and a benefit of \$290,000 upon entry into State 2. The policy is sold to a life in the healthy state (State 0). The interest rate is $\delta = 0.05$

We have ${}_tp_{42}^{00} = e^{-\int_0^t 0.006+0.000006(42+t)\,dt} = e^{-\int_0^t 0.006252+0.000006t\,dt} = e^{-0.006252t-0.000003t^2}$. We therefore calculate

$$\begin{aligned}
\overline{a}_{42}^{00} &= \int_0^\infty e^{-0.05t}e^{-0.006252t-0.000003t^2}\,dt\\
&= \int_0^\infty e^{-0.000003(t+9375.33333333)^2+263.690625333}\,dt\\
&= \sqrt{\frac{\pi}{0.000003}}e^{263.690625333}\left(1-\Phi\left(\sqrt{0.000006}\times 9375.33333333\right)\right)\\
&= 17.74362732
\end{aligned}$$

Next, we can calculate

$$
\begin{aligned}
\overline{A}_{42}^{01} &= \int_0^\infty 0.001e^{-0.05t}e^{-0.006252t-0.000003t^2}\,dt \\
&= 0.001\overline{a}^{00} \\
&= 0.01774362732
\end{aligned}
$$

$$
\begin{aligned}
{}_tp_{42}^{01} &= 0.001\int_{42}^t {}_sp_{42}^{00}\,{}_{t-s}p_{42+s}^{11}\,ds \\
&= 0.001\int_{42}^t e^{-0.006252s-0.000003s^2}e^{-\int_{42+s}^{42+t}0.13\,dx}\,ds \\
&= 0.001\int_{42}^t e^{-0.006252s-0.000003s^2}e^{-(0.13(t-s))}\,ds \\
&= 0.001e^{-0.13t}\int_{42}^t e^{0.123748s-0.000003s^2}\,ds
\end{aligned}
$$

$$
\begin{aligned}
\overline{A}_{42}^{02} &= \int_0^\infty e^{-0.05t}\left({}_tp_{42}^{00}\mu_{42+t}^{02} +_t p_{42}^{01}\mu_{42+t}^{12}\right)\,dt \\
&= \int_0^\infty e^{-0.05t}0.002e^{-0.006252t-0.000003t^2}\,dt \\
&\qquad + \int_0^\infty 0.07e^{-0.05t}\int_0^t 0.001e^{-0.006252s-0.000003s^2}e^{-0.13(t-s)}\,ds\,dt \\
&= 0.03548725464 + 0.00007\int_0^\infty\int_0^\infty e^{-0.05(r+s)}e^{-0.006252s-0.000003s^2}e^{-0.13r}\,dr\,ds \\
&= 0.03548725464 + 0.00007\left(\int_0^\infty e^{-0.18r}\,dr\right)\left(\int_0^\infty e^{-0.056252s-0.000003s^2}\,ds\right) \\
&= 0.03548725464 + 0.00007\times 17.74362732\left[-\frac{e^{-0.18r}}{0.18}\right]_0^\infty \\
&= 0.03548725464 + \frac{0.00007\times 17.74362732}{0.18} \\
&= 0.0423875541533
\end{aligned}
$$

Thus the net annual rate of premium is $\frac{110000\times 0.01774362732+290000\times 0.0423875541533}{17.74362732} =$ \$802.78.

5. *An insurer offers a life insurance policy with an additional benefit for accidental death. The possible exits from this policy are surrender, death (accident) and death (other). The transition intensities are*

$$\mu_x^{01} = 0.06 - 0.001x$$
$$\mu_x^{02} = 0.007 - 0.00004x$$
$$\mu_x^{03} = 0.003 + 0.00008x$$

Calculate the probability that an individual aged 28 surrenders the policy before age 44. [State 0 is in force, State 1 is surrender, State 2 is death (accident) and State 3 is death (other).]

We have

$${}_tp_{28}^{00} = e^{-\int_0^t 0.07-0.00096(28+t)\,dt} = e^{-0.04312t+0.00048t^2}$$

This gives us

$$\begin{aligned}
{}_{16}p_{28}^{01} &= \int_0^{16} (0.06 - 0.001(28+t))e^{-0.04312t+0.00048t^2}\,dt \\
&= \int_0^{16} (0.032 - 0.001t)e^{0.00048(t^2-2\times 44.9166666667t)} \\
&= \int_0^{16} (0.032 - 0.001t)e^{0.00048(t-44.9166666667)^2}e^{-0.968403333336}\,dt \\
&= 0.001e^{-0.968403333336}\int_0^{16} (32-t)e^{0.00048(t-44.9166666667)^2}\,dt \\
&= 0.001e^{-0.968403333336}\int_{28.9166666667}^{44.9166666667} (u-12.9166666667)e^{0.00048u^2}\,du \\
&= 0.2962667
\end{aligned}$$

where the integral is calculated using numerical integration.

Standard Questions

6. *An insurance company is developing a new model for transition intensities in a disability income model. Under these transition intensities it calculates*

$$\begin{array}{lll}
\overline{A}_{47}^{02} = 0.502563 & \overline{A}_{65}^{02} = 0.749001 & \overline{a}_{65}^{11} = 6.318455 \\
\overline{a}_{47}^{00} = 13.204035 & \overline{a}_{65}^{00} = 8.405827 & \overline{a}_{65}^{10} = 1.482474 \\
{}_{18}p_{47}^{00} = 0.803335 & {}_{18}p_{47}^{01} = 0.035908 & \delta = 0.03
\end{array}$$

Calculate the premium for an 18-year policy for a life aged 47, with continuous premiums payable while in the healthy, which pays a continuous benefit while in the sick state, at a rate of \$80,000 per year, and pays a death benefit of \$500,000 immediately upon death.

[Hint: to calculate $\overline{a}_x^{01}$, consider how to extend the equation $\overline{a}_x = \frac{1-\overline{A}_x}{\delta}$ to the multiple state case by combining states 0 and 1.]

The premium is

$$\begin{aligned} P &= \frac{80000\overline{a}^{01}_{47:\overline{18|}} + 500000A^{02}_{47:\overline{18|}}}{\overline{a}^{00}_{47:\overline{18|}}} \\ &= \frac{80000(\overline{a}^{01}_{47} - e^{-0.54}({}_{18}p^{00}_{47}\overline{a}^{01}_{65} +_{18} p^{01}_{47}\overline{a}^{11}_{65})) + 500000A^{02}_{47} - e^{-0.54}({}_{18}p^{00}_{47}\overline{A}^{02}_{65} +_{18} p^{01}_{47}\overline{A}^{12}_{65})}{\overline{a}^{00}_{47} - e^{-0.54}({}_{18}p^{00}_{47}\overline{a}^{00}_{65} +_{18} p^{01}_{47}\overline{a}^{10}_{65})} \end{aligned}$$

We therefore need to calculate all the quantities in this expression. We are given ${}_{18}p^{00}_{47}$, p^{01}_{47}, $\overline{a}^{11}_{65}$, $\overline{A}^{02}_{47}$, $\overline{A}^{02}_{65}$, $\overline{a}^{00}_{47}$, $\overline{a}^{00}_{65}$ and $\overline{a}^{10}_{65}$. We therefore need to calculate $\overline{a}^{01}_{47}$, $\overline{a}^{01}_{65}$ and $\overline{A}^{12}_{65}$.

By combining the states 0 and 1, we get that $\overline{a}^{00}_x + \overline{a}^{01}_x = \frac{1-\overline{A}^{02}_x}{\delta}$. This gives us

$$\overline{a}^{01}_x = \frac{1-\overline{A}^{02}_x}{\delta} - \overline{a}^{00}_x$$

so

$$\begin{aligned} \overline{a}^{01}_{47} &= \frac{1-\overline{A}^{02}_{47}}{\delta} - \overline{a}^{00}_{47} = \frac{1-0.502563}{0.03} - 13.204035 = 3.3771983333 \\ \overline{a}^{01}_{65} &= \frac{1-\overline{A}^{02}_{65}}{\delta} - \overline{a}^{00}_{65} = \frac{1-0.749001}{0.03} - 8.405827 = -0.03919366667 \\ \overline{A}^{12}_{65} &= 1 - \delta(\overline{a}^{10}_{65} + \overline{a}^{11}_{65}) = 1 - 0.03(1.482474 + 6.318455) = 0.76597213 \end{aligned}$$

Substituting these values and the values given into the expression for P gives:

$$
\begin{aligned}
\overline{a}^{01}_{47:\overline{18|}} &= 3.377198 - e^{-0.54}(0.803335 \times (-0.03919366667) + 0.035908 \times 6.318455) \\
&= 3.26333048453 \\
A^{02}_{47:\overline{18|}} &= 0.502563 - e^{-0.54}(0.803335 \times 0.749001 + 0.035908 \times 0.76597213) \\
&= 0.135895908251 \\
\overline{a}^{00}_{47:\overline{18|}} &= 13.204035 - e^{-0.54}(0.803335 \times 8.405827 + 0.035908 \times 1.482474 \\
&= 9.23789252154 \\
P &= \frac{80000 \times 3.26333048453 + 500000 \times 0.135895908251}{9.23789252154} \\
&= \$35,615.7415905
\end{aligned}
$$