ACSC/STAT 4720, Life Contingencies II

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Homework Sheet 3

Model Solutions

Basic Questions

1. *A life aged 62 wants to buy a 5-year term insurance policy. A life-table based on current-year (2021) mortality is:*

x	l_x	d_x
62	10000.00	157.11
63	9842.89	167.55
64	9675.34	178.46
65	9496.87	189.81
66	9307.06	201.57

The insurance company uses a single-factor scale function $q(x,t) = q(x,0)(1-\phi_x)^t$ to model changes in mortality. The insurance company uses the following values for ϕ_x:

x	ϕ_x
62	0.01
63	0.02
64	0.03
65	0.01
66	0.01

Calculate $\ddot{a}_{62:\overline{5}|}$ at interest rate $i = 0.06$, taking into account the change in mortality.

We calculate the following mortalities for this individual:

t	$q(62+t, 2021+t)$	$1-q(62+t, 2021+t)$
0	0.015711	0.984289
1	$\frac{167.55}{9842.89}(1-0.02)^1 = 0.0166819907567$	0.983318009243
2	$\frac{178.46}{9675.34}(1-0.03)^2 = 0.0173547404019$	0.982645259598
3	$\frac{189.81}{9496.87}(1-0.01)^3 = 0.0193929634911$	0.980607036509
4	$\frac{201.57}{9307.06}(1-0.01)^4 = 0.0208043504325$	0.979195649568

This gives us

$$
\begin{aligned}
\ddot{a}_{54:\overline{5}|} &= 1+0.984289\left((1.06)^{-1}+0.983318009243\left((1.06)^{-2}+0.982645259598\left((1.06)^{-3}+0.980607036509(1.06)^{-4}\right)\right)\right)\\
&= 4.32724159115
\end{aligned}
$$

2. *Using the lifetable from Question 1, the insurance company now uses the following mortality scale, $\phi(x,t)$ based on both age and year:*

	t				
x	*2022*	*2023*	*2024*	*2025*	*2026*
62	0.015	0.015	0.020	0.015	0.020
63	0.045	0.000	0.005	0.020	0.015
64	−0.020	0.005	0.005	0.025	0.010
65	0.025	0.005	0.030	0.015	0.010
66	0.025	0.015	0.040	0.010	0.025

Use this mortality scale to calculate $A^{1}_{62:\overline{5}|}$ *at interest rate* $i = 0.05$.

We calculate the following mortalities for this individual:

t	$q(62+t, 2021+t)$	$1-q(62+t, 2021+t)$
0	0.015711	0.984289
1	$\frac{167.55}{9842.89}(1-0.045) = 0.0162564297681$	0.983743570232
2	$\frac{178.46}{9675.34}(1+0.020)(1-0.005) = 0.0187196578105$	0.98128034219
3	$\frac{189.81}{9496.87}(1-0.025)(1-0.005)(1-0.03) = 0.0188078012506$	0.981192198749
4	$\frac{201.57}{9307.06}(1-0.025)(1-0.015)(1-0.04)(1-0.01) = 0.0197679054855$	0.980232094515

This gives us

$$
\begin{aligned}
A^{1}_{54:\overline{5}|} &= 0.015711(1.04)^{-1} + 0.984289\times 0.0162564297681(1.04)^{-2} + 0.984289\times 0.983743570232\times 0.0187196578105(1.04)^{-3}\\
&\quad + 0.984289\times 0.983743570232\times 0.98128034219\times 0.0188078012506(1.04)^{-4}\\
&\quad + 0.984289\times 0.983743570232\times 0.98128034219\times 0.981192198749\times 0.0197679054855(1.04)^{-5}\\
&= 0.0764379496505
\end{aligned}
$$

3. *A life-insurance company has the current mortality scale for 2021:*

x	$\phi(x, 2022)$	$\left.\frac{d\phi(x,t)}{dt}\right\|_{x,t=2022}$	$\left.\frac{d\phi(x+t,t)}{dt}\right\|_{x,t=2022}$
62	*0.019716147074*	0.0010174563604	−0.0072041156604
63	*0.002020553601*	−0.0034265947953	0.0026162132756
64	*0.006613716415*	−0.0003726756896	0.0027308379647
65	*0.007275748793*	0.0002926793710	−0.0009361061799
66	*0.002408521108*	−0.0019393709894	0.0007201245263

Current mortality (in 2021) is given in the lifetable in Question 1. The company assumes that from 2031 onwards, we will have $\phi(x,t) = 0.01$ *for* *all x and t. Calculate $\ddot{a}_{62:\overline{5}|}$ at interest rate $i = 0.05$, using the average of age-based and cohort-based effects.*

If we are given $\phi(x, 2022) = q$, $\phi'(x, 2022) = r$ and $\phi(x, t) = 0.01$ for all x and t from 2031 onwards, we let $f(t) = \phi(x, 2022 + t)$, then we have

$$\begin{aligned} f(0) &= q \\ f'(0) &= r \\ f(12) &= 0.01 \\ f'(12) &= 0 \end{aligned}$$

For cubic interpolation, we have $f(t) = at^3 + bt^2 + ct + d$, so our equations become

$$\begin{aligned} d &= q \\ c &= r \\ 9^3a + 9^2b + 9c + d &= 0.01 \\ 3 \times 9^2a + 2 \times 9b + c &= 0 \\ 27b + 6c + d &= 0.01 \\ b &= \frac{0.01 - 6r - q}{27} \\ a &= \frac{0.01 - 3(0.01 - 6r - q) - 9r - q}{729} \\ &= \frac{-0.02 + 9r + 2q}{729} \end{aligned}$$

Using this, we calculate the coefficients for each age:

x	a	b	c	d
63	−0.0000641951	0.0010570006	−0.0034265947953	0.002020553601
64	−0.0000138912	0.0002082347	−0.0003726756896	0.006613716415
65	−0.0000038606	0.0000358583	0.0002926793710	0.007275748793
66	−0.0000447700	0.0007121372	−0.0019393709894	0.002408521108

This gives us, for example

$$\phi(65, 2024) = -0.0000038606 \times 2^3 + 0.0000358583 \times 2^2 + 0.0002926793710 \times 2 + 0.007275748793 = 0.007973655935$$

Similarly, we compute the following value of $\phi(x, t)$

x	$\phi(x, 2022)$	$\phi(x, 2023)$	$\phi(x, 2024)$	$\phi(x, 2025)$	$\phi(x, 2026)$	
63	0.002020553601	−0.0004132357576	−0.001118194741	−0.000479494097	0.001117695424	0.003288203075
64	0.006613716415	0.0064353843056	0.006590174757	0.006994740869	0.007565735742	0.008219812474
65	0.007275748793	0.0076004258821	0.007973655948	0.008372275304	0.008773120262	0.009153027133
66	0.002408521108	0.0011365173803	0.001020168358	0.001790854316	0.003179955529	0.004918852270

For cohort-based values, if we let $g_x(t) = \phi(x+t, 2021+t) = at^3 + bt^2 + ct + d$, with $\phi(x, 2021) = q$, and $\frac{d\phi(x+t,2021+t)}{dt} = r$, then we have

$$g(0) = q$$
$$g'(0) = r$$
$$g(9) = 0.015$$
$$g'(9) = 0$$

Substituting the cubic interpolation, we get the same equations

$$d = q$$
$$c = r$$
$$b = \frac{0.01 - 6r - q}{27}$$
$$a = \frac{-0.02 + 9r + 2q}{729}$$

Using this, we calculate the coefficients for each cohort:

x	a	b	c	d
62	−0.00006228360329	0.0012410572922	−0.0072041156604	0.019716147074
63	0.00001040744401	−0.0002858456761	0.0026162132756	0.002020553601
64	0.00002442383335	−0.0004814349705	0.0027308379647	0.006613716415
65	−0.00001903080663	0.0003089217884	−0.0009361061799	0.007275748793
66	−0.00001193667633	0.0001211382124	0.0007201245263	0.002408521108

From this we get

$$\phi(64, 2023) = 0.00001040744401 \times 1^3 - 0.0002858456761 \times 1^2 + 0.0026162132756 \times 1 + 0.002020553601 = 0.00436132864451$$

and similarly, we get the following values:

x	$\phi(x, 2022)$	$\phi(x, 2023)$	$\phi(x, 2024)$	$\phi(x, 2025)$	$\phi(x, 2026)$
62	0.019716147074				
63	0.002020553601	0.013690805102			
64	0.006613716415	0.004361328644	0.009773876095		
65	0.007275748793	0.008887543242	0.006192857000	0.007591658433	
66	0.002408521108	0.006629533595	0.010345043129	0.007577583331	0.006770450497

Taking the average of the age-based and cohort-based values of $\phi(x, t)$ gives the following:

x	$\phi(x, 2022)$	$\phi(x, 2023)$	$\phi(x, 2024)$	$\phi(x, 2025)$	$\phi(x, 2026)$
63	0.002020553601				
64	0.006613716415	0.0054757517005			
65	0.007275748793	0.008430599595	0.007282566152		
66	0.002408521108	0.0038248509765	0.0060679487225	0.00537876943	

This gives us

$$q(62,2021) = \frac{157.11}{10000.00} = 0.015711$$
$$q(63,2022) = (1-0.002020553601) \times \frac{167.55}{9842.89} = 0.0169880447962$$
$$q(64,2023) = (1-0.006613716415)(1-0.0054757517005) \times \frac{178.46}{9675.34} = 0.0182225096443$$
$$q(65,2024) = (1-0.007275748793)(1-0.008430599595)(1-0.007282566152) \times \frac{189.81}{9496.87} = 0.0195306183016$$
$$q(66,2025) = (1-0.002408521108)(1-0.0038248509765)(1-0.0060679487225)(1-0.00537876943) \times \frac{201.57}{9307.06} = 0.0212772862894$$

and therefore

$$\begin{aligned}\ddot{a}_{62:\overline{5}|} &= 1 + (1-0.015711)(1.04^{-1} + (1-0.016988)(1.04^{-2} + (1-0.018223)(1.04^{-3} + (1-0.019531)1.04^{-4}))) \\ &= 4.48164306474\end{aligned}$$

Standard Questions

4. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.65 \qquad \sigma_k = 1.4 \qquad K_{2021} = -3.29$$

And the following values of α_x and β_x:

x	α_x	β_x
42	−3.445547529	*0.2160196693*
43	−3.723003508	*0.2056043631*
44	−3.240526315	*0.2319018119*
45	−3.213960546	*0.2160218805*
46	−3.394213139	*0.2669114067*
47	−3.014411418	*0.2324790526*
48	−3.275815282	*0.2361910612*

The insurance company simulates the following values of Z_t:

-0.8654056910 -0.9142362784 -1.2831326166 1.0005379227 0.3053339512 0.1684182795 -0.1596511482

Using these simulated values, calculate the probability that a life aged exactly 42 at the start of 2021 survives for 6 years.

Using our simulated values of Z_t, we simulate

$$K_{2022} = K_{2021} - 0.65 + 1.4 \times -0.8654056910 = -5.1515679674$$
$$K_{2023} = K_{2022} - 0.65 + 1.4 \times -0.9142362784 = -7.08149875716$$
$$K_{2024} = K_{2023} - 0.65 + 1.4 \times -1.2831326166 = -9.5278844204$$
$$K_{2025} = K_{2024} - 0.65 + 1.4 \times 1.0005379227 = -8.77713132862$$
$$K_{2026} = K_{2025} - 0.65 + 1.4 \times 0.3053339512 = -8.99966379694$$
$$K_{2027} = K_{2026} - 0.65 + 1.4 \times 0.1684182795 = -9.41387820564$$
$$K_{2028} = K_{2027} - 0.65 + 1.4 \times -0.1596511482 = -10.2873898131$$

This gives us

$$\log(m(42, 2021)) = -3.445547529 + 0.2160196693 \times -3.29 = -4.156252241$$
$$\log(m(43, 2022)) = -3.723003508 + 0.2056043631 \times -5.1515679674 = -4.7821883589$$
$$\log(m(44, 2023)) = -3.240526315 + 0.2319018119 \times -7.08149875716 = -4.88273870775$$
$$\log(m(45, 2024)) = -3.213960546 + 0.2160218805 \times -9.5278844204 = -5.27219205568$$
$$\log(m(46, 2025)) = -3.394213139 + 0.2669114067 \times -8.77713132862 = -5.73692960871$$
$$\log(m(47, 2026)) = -3.014411418 + 0.2324790526 \times -8.99966379694 = -5.10664473123$$
$$\log(m(48, 2027)) = -3.275815282 + 0.2361910612 \times -9.41387820564 = -5.4992891654$$
$$m(42, 2021) = 0.0156661610322$$
$$m(43, 2022) = 0.00837764559763$$
$$m(44, 2023) = 0.00757623649117$$
$$m(45, 2024) = 0.00513234784104$$
$$m(46, 2025) = 0.00322465404064$$
$$m(47, 2026) = 0.00605636961875$$
$$m(48, 2027) = 0.00408967748974$$

Using the approximation $q(x, t) \approx 1 - e^{-m(x,t)}$, we get

$$
\begin{aligned}
q(42, 2021) &\approx 0.01554408505 \\
q(43, 2022) &\approx 0.008342650917 \\
q(44, 2023) &\approx 0.007547609153 \\
q(45, 2024) &\approx 0.005119199847 \\
q(46, 2025) &\approx 0.003219460428 \\
q(47, 2026) &\approx 0.00603806678 \\
q(48, 2027) &\approx 0.004081326147
\end{aligned}
$$

The probability that the life survives 6 years is therefore

$$(1-0.01554408505)(1-0.008342650917)(1-0.007547609153)(1-0.005119199847)(1-0.003219460428)(1-0.00603806678) = 0.95501005035$$