ACSC/STAT 4720, Life Contingencies II

FALL 2021
Toby Kenney

Homework Sheet 4

Model Solutions

1. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.3 \qquad \sigma_k = 1.2 \qquad K_{2021} = -2.25 \qquad \alpha_{36} = -1.74 \qquad \beta_{36} = 1.11$$

It estimates that its reserves are adequate in a given year provided $q(36, t) < 0.0034$. Calculate the probability that its reserves are still adequate in 7 years' time. Use UDD to calculate the relation between q_x and m_x.

Recall that $m_x = \frac{q_x}{\int_0^1 {}_tp_x\,dt}$. Under UDD, we have

$$\int_0^1 {}_tp_x\,dt = \int_0^1 (1 - tq_x)\,dt = \left[t - q_x\frac{t^2}{2}\right]_0^1 = 1 - \frac{q_x}{2}$$

Therefore, we have that $q(36, t) < 0.0034$ if and only if $m(34, t) < \frac{0.0034}{0.9983} = 0.00340578984273$.

The Lee-Carter model gives $\log(m(34, 2028)) = \alpha_{34} + \beta_{34}K_{2028} = -1.74 + 1.11K_{2028}$. We therefore have $m(34, t) < 0.00340578984273$ if

$$\begin{aligned} -1.74 + 1.11K_{2028} &< \log(0.00340578984273) = -5.68227840072 \\ K_{2028} &< -3.55160216281 \end{aligned}$$

We have $K_{2028} = K_{2021} + 7c + \sigma_k(Z_{2022} + Z_{2023} + Z_{2024} + Z_{2025} + Z_{2026} + Z_{2027} + Z_{2028}) = -2.25 - 2.1 + 1.2(Z_{2022} + Z_{2023} + Z_{2024} + Z_{2025} + Z_{2026} + Z_{2027} + Z_{2028}))$. Therefore we have $K_{2028} < -3.55160216281$ if and only if $Z_{2022} + Z_{2023} + Z_{2024} + Z_{2025} + Z_{2026} + Z_{2027} + Z_{2028} < \frac{-3.55160216281-(-4.35)}{1.2} = 0.665331530992$. Since each Z_t is i.i.d. standard normal. $Z_{2022}+Z_{2023}+Z_{2024}+Z_{2025}+Z_{2026}+Z_{2027}+Z_{2028}$ is normal with mean 0 and variance 7, so $P(Z_{2022}+Z_{2023}+Z_{2024}+Z_{2025}+Z_{2026}+Z_{2027}+Z_{2028} < 0.665331530992) = \Phi\left(\frac{0.665331530992}{\sqrt{7}}\right) = \Phi(0.251471681487) = 0.599275273117$.

2. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$$K^{(1)}_{2021} = -9.37 \qquad K^{(2)}_{2021} = 0.11 \qquad c^{(1)} = -0.12 \qquad c^{(2)} = 0.02$$
$$\sigma_{k_1} = 0.7 \qquad \sigma_{k_2} = 0.06 \qquad \rho = 0.4 \qquad \overline{x} = 46$$

(a) Use this scale to calculate the median value of $q(33, 2029)$.

Under the CBD model, we have $\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right) = K^{(1)}_{2029} + K^{(2)}_{2029}(33-46)$. Since $\log\left(\frac{q(33,2029)}{1-q(33,2029)}\right)$ is an increasing function of $q(33, 2029)$, the median value of $q(33, 2029)$ corresponds to the median value of $K^{(1)}_{2029} + K^{(2)}_{2029}(33-46) = K^{(1)}_{2029} - 13K^{(2)}_{2029}$. $K^{(1)}_{2029}$ is normally distributed with mean $K^{(1)}_{2021} + 8c^{(1)}$, and $K^{(2)}_{2029}$ has mean $K^{(2)}_{2021} + 8c^{(2)}$. Therefore the mean of $K^{(1)}_{2029} - 13K^{(2)}_{2029}$ is $K^{(1)}_{2021} + 8c^{(1)} - 13(K^{(2)}_{2021} + 8c^{(2)}) = -9.37 + 8 \times (-0.12) - 13(0.11 + 8 \times 0.02) = -13.84$. Since $K^{(1)}_{2029} - 13K^{(2)}_{2029}$ is normally distributed, the mean is the median, so the median value of $q(33, 2029)$ is $\frac{e^{-13.84}}{1+e^{-13.84}} = 9.75807039208 \times 10^{-7}$.

(b) A life aged 72 will only be approved for life insurance if her mortality is less than 0.1. How long can she wait to purchase a life insurance contract and still have a 70% probability of being approved? [Remember that her age also increases by 1 each year.]

If $q(x,t) < 0.1$, we have $\log\left(\frac{q(x,t)}{1-q(x,t)}\right) < \log\left(\frac{0.1}{0.9}\right) = -2.19722457734$, so we want to find the probability that $K^{(1)}_{2021+t} + (26+t)K^{(2)}_{2021+t} < -2.19722457734$.

We have that $K^{(1)}_{2021+t} = -9.37 - 0.12t + 0.7(Z^{(1)}_{2022} + \cdots + Z^{(1)}_{2021+t})$ and $K^{(2)}_{2021+t} = 0.11 + 0.02t + 0.7(Z^{(1)}_{2022} + \cdots + Z^{(1)}_{2021+t})$ are both normally distributed. Therefore, $K^{(1)}_{2021+t} + (26+t)K^{(2)}_{2021+t} = -9.37 - 0.12t + 0.7(Z^{(1)}_{2022} + \cdots + Z^{(1)}_{2021+t}) + (26+t)(0.11 + 0.02t + 0.06(Z^{(1)}_{2022} + \cdots + Z^{(1)}_{2021+t}))$ $K^{(1)}_{2021+t} + (26+t)K^{(2)}_{2021+t} = -6.51 + 0.51t + 0.02t^2 + 0.7(Z^{(1)}_{2022} + \cdots + Z^{(1)}_{2021+t}) + (26+t)0.06(Z^{(2)}_{2022} + \cdots + Z^{(2)}_{2021+t}))$ This has mean $-6.51 + 0.51t + 0.02t^2$ and variance $t(0.7^2 + 0.06^2(26+t)^2 + 2 \times 0.7 \times 0.06(26+t) \times 0.4) = 3.7972t + 0.2208t^2 + 0.0036t^3$

The probability that she is approved in t year's time is therefore $\Phi\left(\frac{6.51-0.51t-0.02t^2-2.19722457734}{\sqrt{3.7972t+0.2208t^2+0.0036t^3}}\right)$. This probability is more than 70% provided

$$\Phi\left(\frac{-6.51 + 0.51t + 0.02t^2 + 2.19722457734}{\sqrt{3.7972t + 0.2208t^2 + 0.0036t^3}}\right) < 0.3$$
$$\frac{-4.31277542266 + 0.51t + 0.02t^2}{\sqrt{3.7972t + 0.2208t^2 + 0.0036t^3}} < -0.524400512708$$

Numerically, we see that this is first satisfied for $t < 4$ years, so she can wait for 3 years.

Standard Questions

3. *An insurance company uses a Lee-Carter model and fits the following parameters:*

$$c = -0.5 \qquad \sigma_k = 1.3 \qquad K_{2021} = -5.12$$

And the following values of α_x and β_x:

x	α_x	β_x
33	-6.788236	*0.2228085*
34	-6.750172	*0.1375526*
35	-6.755374	*0.1979110*
36	-6.720697	*0.1529246*
37	-6.694897	*0.2131581*

Using the approximation $m(x,t) \approx q(x,t)$, calculate the probability that a life aged 33 dies at age 35 under this model.

The probability that the life dies aged 35 is $\mathbb{E}\left((1-q(33,2021))(1-q(34,2022))q(35,2023))\right)$ Using the approximation $m(x,t) \approx q(x,t)$, we have $\log(q(x,t)) = \alpha_x + \beta_x K_t$. We have that

$$\begin{aligned}
\log(q(33,2021)) &= -6.788236 + 0.2228085 \times -5.12 = -7.92901552 \\
\log(q(34,2022)) &= -6.750172 + 0.1375526 \times (-5.12 - 0.5 + 1.3Z_{2022}) \\
&= -7.523217612 + 0.17881838Z_{2022}) \\
\log(q(35,2023)) &= -6.755374 + 0.1979110 \times (-5.12 - 1.0 + 1.3(Z_{2022} + Z_{2023})) \\
&= -7.96658932 + 0.2572843(Z_{2022} + Z_{2023}))
\end{aligned}$$

The probability we want to calculate is

$$\begin{aligned}
&\mathbb{E}\left((1-q(33,2021))(1-q(34,2022))q(35,2023))\right) \\
&= (1-q(33,2021))\mathbb{E}\left((1-q(34,2022))q(35,2023)\right) \\
&= (1-q(33,2021))\left(\mathbb{E}q(35,2023) - \mathbb{E}(q(34,2022)q(35,2023))\right) \\
=&0.99963985921\left(\mathbb{E}q(35,2023) - \mathbb{E}(q(34,2022)q(35,2023))\right)
\end{aligned}$$

We know that $q(35,2023)$ is log-normal with parameters $\mu = -7.96658932$ and $\sigma^2 = 2 \times 0.2572843^2 = 0.132390422053$. Furthermore

$\log(q(34,2022)q(35,2023)) = \log(q(34,2022))+\log(q(35,2023)) = (-7.523217612+0.17881838Z_{2022})+(-7$

so $q(34,2022)q(35,2023)$ is log-normal with $\mu = -15.489806932$ and $\sigma^2 = 0.43610268^2 + 0.2572843^2 = 0.25638075853$ This gives $\mathbb{E}(q(35,2023)) = e^{-7.96658932+\frac{0.132390422053}{2}} = 0.000370597455893$. Similarly $\mathbb{E}(q(34,2022)q(35,2023)) = e^{-15.489806932+\frac{0.25638075853}{2}} = 2.13076079832 \times 10^{-7}$. Thus, the probability that the life dies aged 35 is

$$0.99963985921\left(0.000370597455893 - 2.13076079832 \times 10^{-7}\right) = 0.00037025098929$$

4. *An insurance company uses a Cairns-Blake-Dowd model with the following parameters:*

$$\begin{array}{llll} K^{(1)}_{2021} = -4.33 & K^{(2)}_{2021} = 0.22 & c^{(1)} = -0.15 & c^{(2)} = 0.01 \\ \sigma_{k_1} = 0.8 & \sigma_{k_2} = 0.06 & \overline{x} = 53 & \end{array}$$

It has not yet decided on a suitable value of ρ. The company sells both life insurance and annuity contracts. It's expected profit in 2024 (in millions) is $36+22.4q(58,2024)-38.8q(73,2024)$. In order to satisfy the regulators, it needs to ensure that the expected profit has a 95% probability of being positive. For what values of ρ is this achieved?

(i) $\rho < 0.24$

(ii) $\rho > 0.24$

(iii) $\rho < 0.33$

(iv) $\rho > 0.33$

(v) $\rho < 0.45$

(vi) $\rho > 0.45$

(vii) $\rho < 0.52$

(viii) $\rho > 0.52$

Justify your answer. [You may need to use simulation to numerically estimate the probability of profit.]

We have that $\log\left(\frac{q(58,2024)}{1-q(58,2024)}\right) = K^{(1)}_{2024}+5K^{(2)}_{2024}$ and $\log\left(\frac{q(73,2024)}{1-q(73,2024)}\right) = K^{(1)}_{2024}+20K^{(2)}_{2024}$ Recalling that $K^{(1)}_{2024} = K^{(1)}_{2021}+3c^{(1)}+\sigma_{k_1}(Z^{(1)}_{2022}+Z^{(1)}_{2023}+Z^{(1)}_{2024}) = -4.78+0.8(Z^{(1)}_{2022}+Z^{(1)}_{2023}+Z^{(1)}_{2024})$ and $K^{(2)}_{2024} = K^{(2)}_{2024}+3c^{(2)}+\sigma_{k_1}(Z^{(1)}_{2022}+Z^{(1)}_{2023}+Z^{(1)}_{2024}) = 0.25+0.06(Z^{(1)}_{2022}+Z^{(1)}_{2023}+Z^{(1)}_{2024})$ We have that $\sigma_{k_1}Z^{(1)}_t + a\sigma_{k_2}Z^{(2)}_t$ is normal with mean 0 and variance ${\sigma_{k_1}}^2 + a^2{\sigma_{k_2}}^2 + 2a\rho\sigma_{k_1}\sigma_{k_2} = 0.64+0.0036a^2+0.096\rho a$, so $\sum_{t=2022}^{2024}\sigma_{k_1}Z^{(1)}_t + a\sigma_{k_1}Z^{(2)}_t$ is normal with mean 0 and variance $3(0.64+0.0036a^2+0.096\rho a) = 1.92+0.0108a^2+0.288\rho a$ Finally, the covariance of $\sum_{t=2021}^{2024}\sigma_{k_1}Z^{(1)}_t + a_1\sigma_{k_1}Z^{(2)}_t$ and $\sum_{t=2021}^{2024}\sigma_{k_1}Z^{(1)}_t + a_2\sigma_{k_1}Z^{(2)}_t$ is

$$\frac{1}{2}\left(\text{Var}\left(\sum_{t=2022}^{2024} 2\sigma_{k_1} Z_t^{(1)} + (a_1+a_2)\sigma_{k_2} Z_t^{(2)}\right) - \text{Var}\left(\sum_{t=2022}^{2024} \sigma_{k_1} Z_t^{(1)} + a_1\sigma_{k_2} Z_t^{(2)}\right) - \text{Var}\left(\sum_{t=2022}^{2024} \sigma_{k_1} Z_t^{(1)} + a_2\sigma_{k_2} Z_t^{(2)}\right)\right)$$

$$\frac{1}{2}\left(4\left(1.92 + 0.0108\left(\frac{a_1+a_2}{2}\right)^2 + 0.288\rho\frac{(a_1+a_2)}{2}\right) - (1.92 + 0.0108a_1^2 + 0.288\rho a_1) - (1.92 + 0.0108a_2^2 + 0.288\rho a_2)\right)$$

$$= 1.92 + 0.0108a_1a_2 + 0.288\rho\left(\frac{a_1+a_2}{2}\right)$$

In particular, we have that $\log\left(\frac{q(58,2024)}{1-q(58,2024)}\right)$ and $\log\left(\frac{q(73,2024)}{1-q(73,2024)}\right)$ jointly follow a multivariate normal distribution with mean $-3.53, 0.22$ and covariance matrix

$$\begin{pmatrix} 2.19 + 1.44\rho & 3 + 3.6\rho \\ 3 + 3.6\rho & 6.24 + 5.76\rho \end{pmatrix}$$

We use a simulation to estimate the probability that $36 + 22.4q(58, 2024) - 38.8q(73, 2024) > 0$ for different values of ρ.

```
library(MASS)

HW4Q4evaluateProb<-function(rho){
# This function doesn't work with vector inputs!!!!
    a<-mvrnorm(1000000,c(-3.53,0.22),rbind(c(2.19+1.4*rho,3+3.6*rho),c(3+3.6*rho,6.24+5.76*rho)))
ea<-exp(a)
q<-ea/(1+ea)
return(mean(q%*%c(22.4,-38.8)>-36))
}
```

My simulation gives the following probabilities:

ρ	$P(36 + 22.4q(58, 2024) - 38.8q(73, 2024) > 0)$
0.24	0.955233
0.33	0.949648
0.45	0.943467
0.52	0.939586

We see that the probability of a profit is a decreasing function of ρ, and that it is 0.95 when $\rho = 0.33$. Thus the condition is achieved for $\rho < 0.33$.