ACSC/STAT 4720, Life Contingencies II

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Homework Sheet 5

Model Solutions

Basic Questions

1. *A disability income insurance company collects the following claim data (in thousands):*

i	d_i	x_i	u_i	i	d_i	x_i	u_i	i	d_i	x_i	u_i
1	0.0	1.8	-	8	0.0	-	10	15	0.2	2.0	-
2	0.0	2.1	-	9	0.1	1.2	-	16	0.2	2.2	-
3	0.0	2.9	-	10	0.1	2.6	-	17	0.4	3.3	-
4	0.0	3.4	-	11	0.1	3.2	-	18	0.8	-	5
5	0.0	4.0	-	12	0.1	3.5	-	19	0.9	-	10
6	0.0	-	5	13	0.1	4.6	-	20	1.4	7.9	-
7	0.0	-	10	14	0.1	8.3	-	21	1.9	-	5

Using a Kaplan-Meier product-limit estimator:

(a) estimate the probability that a random loss exceeds 2.7.

We record the following values:

x	r_x	d_x
1.2	19	1
1.8	19	1
2.0	19	1
2.1	18	1
2.2	17	1
2.6	16	1
2.9	15	1
3.2	14	1
3.3	13	1
3.4	12	1
3.5	11	1
4.0	10	1
4.6	9	1
7.9	5	1
8.3	4	1

The probability that a random loss exceeds 2.7 is given by

$$\frac{18}{19} \times \frac{18}{19} \times \frac{18}{19} \times \frac{17}{18} \times \frac{16}{17} \times \frac{15}{16} = 0.708558098848$$

(b) estimate the 60th percentile of the distribution.

We compute the cumulative products until the probability becomes less than 0.4. This happens at

$$\frac{18}{19}\times\frac{18}{19}\times\frac{18}{19}\times\frac{17}{18}\times\frac{16}{17}\times\frac{15}{16}\times\frac{14}{15}\times\frac{13}{14}\times\frac{12}{13}\times\frac{11}{12}\times\frac{10}{11}\times\frac{9}{10}\times\frac{8}{9} = 0.377897652719$$

so the 60th percentile is 4.6.

(c) Use a Nelson-Åalen estimator to estimate the 60th percentile of the distribution.

The 60th percentile of the distribution is the first value of x such that $H(x) \geqslant \log(0.4^{-1}) = 0.916290731874$. We compute values until we reach this value:

x	r_x	d_x	$H(x)$
1.2	19	1	$\frac{1}{19} = 0.052631578947$
1.8	19	1	$\frac{1}{19} + \frac{1}{19} = 0.105263157895$
2.0	19	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} = 0.157894736842$
2.1	18	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} = 0.213450292398$
2.2	17	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} = 0.272273821809$
2.6	16	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} = 0.334773821809$
2.9	15	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} = 0.401440488476$
3.2	14	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{1}{14} = 0.472869059905$
3.3	13	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{1}{14} + \frac{1}{13} = 0.549792136828$
3.4	12	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{1}{14} + \frac{1}{13} + \frac{1}{12} = 0.633125470161$
3.5	11	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{1}{14} + \frac{1}{13} + \frac{1}{12} + \frac{1}{11} = 0.72403456107$
4.0	10	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{1}{14} + \frac{1}{13} + \frac{1}{12} + \frac{1}{11} + \frac{1}{10} = 0.82403456107$
4.6	9	1	$\frac{1}{19} + \frac{1}{19} + \frac{1}{19} + \frac{1}{18} + \frac{1}{17} + \frac{1}{16} + \frac{1}{15} + \frac{1}{14} + \frac{1}{13} + \frac{1}{12} + \frac{1}{11} + \frac{1}{10} + \frac{1}{9} = 0.935145672181$

So the Nelson-Åalen estimator for the 60th percentile is 4.6.

2. *For the data in Question 1, use Greenwood's approximation to obtain a 95% confidence interval for the probability that a random loss exceeds 2.7, based on the Kaplan-Meier estimator.*

   (a) Using a normal approximation

   Greenwood's approximation gives

$$\begin{aligned}\mathrm{Var}(S_n(2.7)) &= (0.708558098848)^2\left(\frac{1}{19\times 18}+\frac{1}{19\times 18}+\frac{1}{19\times 18}+\frac{1}{18\times 17}+\frac{1}{17\times 16}+\frac{1}{16\times 15}\right)\\ &= 0.00998237175502\end{aligned}$$

The normal confidence interval is therefore

$0.708558098848 \pm 1.96\sqrt{0.00998237175502} = [0.512730931851, 0.904385265845]$

(b) Using a log-transformed confidence interval.

We have that $\mathrm{Var}(\log(-\log(S_n(2.7)) = \frac{\mathrm{Var}(S_n(2.7))}{S_n(2.7)^2\log(S_n(2.7))^2} = \frac{0.00998237175502}{0.708558098848^2\log(0.708558098848)^2} = 0.167511956051$. We calculate $U = e^{-1.96\sqrt{0.167511956051}} = 0.448344575641$
The log-transformed confidence interval is $[0.708558098848^{\frac{1}{0.448344575641}}, 0.708558098848^{0.448344575641}] = [0.463738709672, 0.856873381814]$.

3. *An insurance company records the following data in a mortality study:*

entry	*death*	*exit*	*entry*	*death*	*exit*	*entry*	*death*	*exit*
66.9	*75.1*	-	*72.2*	*81.1*	-	*74.1*	-	*74.7*
72.5	-	*74.1*	*73.3*	-	*80.3*	*69.3*	*75.8*	-
73.7	-	*76.2*	*72.2*	*74.3*	-	*74.1*	-	*75.6*
72.0	-	*75.6*	*72.5*	*74.6*	-	*74.5*	-	*80.3*
70.5	-	*84.6*	*73.7*	-	*78.9*	*74.2*	-	*74.5*
74.7	-	*75.0*	*74.1*	*76.7*	-	*72.9*	*81.8*	-
67.7	-	*76.3*	*71.9*	-	*74.4*	*74.6*	*74.9*	-
71.6	*76.6*	-	*74.6*	-	*85.1*	*73.4*	-	*75.2*
72.1	-	*74.7*	*74.2*	*79.7*	-	*74.8*	-	*75.5*
71.8	-	*74.1*	*74.9*	-	*76.4*	*66.8*	-	*74.8*
71.4	*76.2*	-	*68.0*	-	*74.3*	*74.8*	-	*82.9*
69.9	-	*76.0*	*73.4*	-	*74.8*	*65.6*	*76.4*	-

Estimate the probability of an individual currently aged exactly 74 dying within the next year using:

(a) the exact exposure method.

The exact exposure is $1+0.1+1+1+1+0.3+1+1+0.7+0.1+1+1+1+1+0.3+0.6+1+0.9+0.4+0.4+0.8+0.1+0.3+0.8+0.6+1+0.9+0.5+0.3+1+0.3+1+0.2+0.8+0.2+1 = 24.6$

The hazard rate is $\frac{3}{24.6} = 0.121951219512$. The probability of dying is $1 - e^{-\frac{3}{24.6}} = 0.114808452475$.

(b) the actuarial exposure method.

The actuarial exposure is $1+0.1+1+1+1+0.3+1+1+0.7+0.1+1+1+1+1+1+1+1+0.9+0.4+0.4+0.8+0.1+0.3+0.8+0.6+1+0.9+0.5+0.3+1+0.4+1+0.2+0.8+0.2+1=25.8$ The probability of dying is therefore $\frac{3}{25.8}=0.116279069767$.

4. *Using the following table:*

Age	*No. at start*	*enter*	*die*	*leave*	*No. at next age*
82	*38*	*41*	*8*	*24*	*47*
83	*47*	*20*	*10*	*24*	*33*
84	*33*	*17*	*11*	*18*	*21*
85	*21*	*11*	*10*	*14*	*8*
86	*8*	*9*	*8*	*6*	*3*

Estimate the probability that an individual aged 83 withdraws from the policy within the next year, conditional on surviving to the end of the year.

Assuming that events happen in the middle of the year, and treating withdrawl as the decrement of interest, the actuarial exposure at age 83 is $47+\frac{20}{2}-\frac{10}{2}=52$. The probability of withdrawl is therefore $\frac{24}{52}=0.461538461538$.

Using the exact exposure method, the exposure is $47+\frac{20}{2}-\frac{10}{2}-\frac{24}{2}=40$, so the probability of withdrawl is $1-e^{-\frac{24}{40}}=0.451188363906$.

5. *In a mortality study of 40 individuals in a disability income policy, an insurance company observes the following transitions, where state H is healthy, D is disabled, S is surrendered and X is dead.*

Entry	*State*	*Time*	*State*	*Time*	*State*	*Exit*	*Entry*	*State*	*Time*	*State*	*Exit*		
46.0	*D*	*46.3*	*S*			*46.3*	*46.0*	*D*	*46.1*	*S*			*46.1*
46.0	*H*					*47.0*	*46.0*	*D*					*47.0*
46.0	*H*					*47.0*	*46.0*	*H*					*47.0*
46.0	*H*	*46.5*	*D*	*47.0*	*H*	*47.0*	*46.0*	*D*	*46.3*	*X*			*47.0*
46.0	*H*	*46.2*	*D*			*47.0*	*46.2*	*H*	*46.5*	*D*	*46.6*	*X*	*46.6*
46.0	*H*	*46.7*	*X*			*46.7*	*46.3*	*H*					*47.0*
46.0	*H*					*47.0*	*46.5*	*H*					*47.0*
46.0	*H*					*47.0*	*46.5*	*H*	*46.9*	*D*			*47.0*
46.0	*H*					*47.0*	*46.5*	*D*					*47.0*
46.0	*H*	*46.2*	*S*			*46.2*	*46.6*	*H*	*47.0*	*D*			*47.0*
46.0	*H*					*47.0*	*46.7*	*D*					*47.0*
46.0	*H*					*47.0*	*46.8*	*H*					*47.0*
46.0	*H*	*46.8*	*S*			*46.8*	*46.8*	*H*					*47.0*
46.0	*H*	*46.6*	*D*			*47.0*	*46.8*	*D*					*47.0*
46.0	*D*	*46.6*	*S*			*46.6*	*46.8*	*H*	*46.8*	*D*			*47.0*
46.0	*H*	*46.2*	*S*			*46.2*	*46.9*	*D*	*46.9*	*S*			*46.9*
46.0	*D*					*47.0*	*46.9*	*H*					*47.0*
46.0	*H*					*47.0*	*46.9*	*H*					*47.0*
46.0	*D*					*47.0*	*47.0*	*D*	*47.0*	*H*			*47.0*
46.0	*D*	*46.3*	*S*			*46.3*	*47.0*	*H*					*47.0*

Based on these data, estimate the probability that an individual aged 46.4 who is disabled becomes healthy and later dies before reaching age 47.

We estimate transition intensities using exact exposure. The exposure in the healthy state is

$0+1+1+0.5+0.2+0.7+1+1+1+0.2+1+1+0.8+0.6+0+0.2+0+1+0+0+0+0+1+0+0.3+0.7+0.5+0.4+0+0.4+0+0.2+0.2+0+0+0+0.1+0.1+0+0=15.1$

The transition intensities from healthy are: surrender — $\frac{3}{15.1}=0.198675496689$; disablility — $\frac{7}{15.1}=0.46357615894$; and death — $\frac{1}{15.1}=0.0662251655629$.

The exposure in the disabled state is $0.3+0+0+0.5+0.8+0+0+0+0+0+0+0+0+0.4+0.6+0+1+0+1+0.3+0.1+1+0+0.3+0.1+0+0+0.1+0.5+0+0.3+0+0+0.2+0.2+0+0+0+0+0=7.7$.

The transition intensities from disabled are: surrender — $\frac{5}{7.7}=0.649350649351$; healthy — $\frac{2}{7.7}=0.25974025974$; and death — $\frac{2}{7.7}=0.25974025974$.

The probability that an individual who is disabled at age 46.4 becomes healthy and then dies before age 47.0 is therefore

$$\begin{aligned}
&\int_0^{0.6} 0.25974025974e^{-1.16883116883s}\int_s^{0.6} 0.0662251655629e^{-0.728476821192(t-s)}\,dt\,ds\\
=&\int_0^{0.6} 0.25974025974e^{-1.16883116883s}\frac{0.0662251655629}{0.728476821192}(1-e^{-0.728476821192(0.6-s)})\,ds\\
=&0.0236127508855\int_0^{0.6}(e^{-1.16883116883s}-e^{-0.437086092715}e^{-0.440354347638s})\,ds\\
=&\frac{0.0236127508855}{1.16883116883}(1-e^{-1.16883116883\times 0.6})-\frac{0.0236127508855e^{-0.437086092715}}{0.440354347638}(1-e^{-0.440354347638\times 0.6})\\
=&0.00214102995367
\end{aligned}$$

Standard Questions

6. *For the study in Question 3, use the exact exposure method, and assume that the number of deaths follows a negative binomial distribution with $r = 1$ and β equal to the exposure multiplied by probability of dying, to find a 95% confidence interval for q_{74}.*

The exposure is 24.6, so the number of deaths follows a negative binomial distribution with $r = 1$ and $\beta = 24.6q_{74}$. Therefore to obtain a 95% confidence interval, find the values of β, such that an observation of 3 is not significant at that level. This means $P(X \geqslant 3) < 0.025$ and $P(X \leqslant 3) < 0.025$. For $r = 1$, the probability mass function given by $P(X = i) = \frac{1}{1+\beta}\left(\frac{\beta}{1+\beta}\right)^i$. We solve

$$\begin{aligned}
P(X<3)=\frac{1}{1+\beta}\left(1+\left(\frac{\beta}{1+\beta}\right)+\left(\frac{\beta}{1+\beta}\right)^2\right)&=0.975\\
(1+\beta)^2+\beta(1+\beta)+\beta^2&=0.975(1+\beta)^3\\
(1+\beta)^3-\beta^3&=0.975(1+\beta)^3\\
\left(\frac{\beta}{1+\beta}\right)^3&=0.025\\
\beta&=0.413231355031
\end{aligned}$$

and

$$
\begin{aligned}
P(X \leqslant 3) = \frac{1}{1+\beta}\left(1+\left(\frac{\beta}{1+\beta}\right)+\left(\frac{\beta}{1+\beta}\right)^2+\left(\frac{\beta}{1+\beta}\right)^3\right) &= 0.025 \\
(1+\beta)^3+\beta(1+\beta)^2+\beta^2(1+\beta)+\beta^3 &= 0.025(1+\beta)^4 \\
(1+\beta)^4-\beta^4 &= 0.025(1+\beta)^4 \\
\left(\frac{\beta}{1+\beta}\right)^4 &= 0.975 \\
\beta &= 157.492088176
\end{aligned}
$$

Therefore the 95% confidence interval for β is $[0.413231355031, 157.492088176]$, and the 95% confidence interval for force of mortality is $\left[\frac{0.413231355031}{24.6}, \frac{157.492088176}{24.6}\right] = [0.0167980225622, 6.40211740553]$. This gives a 95% confidence interval for q_{74} as $\left[1-e^{-0.0167980225622}, 1-e^{-6.40211740553}\right] = [0.016657722468, 0.998341957195]$.