ACSC/STAT 4720, Life Contingencies II

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Homework Sheet 7

Model Solutions

Basic Questions

1. *An insurance company sells a 5-year life insurance policy to a life aged 36, for whom the lifetable below is appropriate.*

x	l_x	d_x
52	*10000.00*	*72.24*
53	*9927.76*	*81.27*
54	*9846.50*	*91.37*
55	*9755.12*	*102.66*
56	*9652.47*	*115.23*
57	*9537.24*	*129.19*
58	*9408.05*	*144.64*

The annual gross premium is \$4,728.60. Initial expenses are \$6,090 plus 20% of the first premium. The death benefits are \$300,000. Renewal costs are 4% of each subsequent premium. The interest rate is $i = 0.03$

(a) Calculate the expected net cash-flows associated with this policy (assuming no reserve). [This is the profit vector for the policy.]

We conduct the following profit test:

Year	Premium	Expenses	Interest	Expected Death Benefits	$\Pr_t$
0		7035.72			−7035.72
1	4,728.60	0	141.858	2167.2	2703.258
2	4,728.60	189.144	136.18368	2455.84099535	2219.79868465
3	4,728.60	189.144	136.18368	2783.83181841	1891.80786159
4	4,728.60	189.144	136.18368	3157.11134256	1518.52833744
5	4,728.60	189.144	136.18368	3581.36311224	1094.27656776

(b) Which of the following is the internal rate of return of the policy:

(i) $i = 0.0962$

(ii) $i = 0.1201$

(iii) $i = 0.1449$

(iv) $i = 0.3066$

We first compute the profit signature

Year	$\Pr_t$	P(in force)	Π_t
0	−7035.72	1	−7035.72
1	2703.258	1	2703.258
2	2219.79868465	0.992776	2203.76285895
3	1891.80786159	0.984650	1862.76861091
4	1518.52833744	0.975512	1481.34261551
5	1094.27656776	0.965247	1056.2471742

The NPV at interest rate i is therefore

$$2703.258(1+i)^{-1}+2203.76285895(1+i)^{-2}+1862.76861091(1+i)^{-3}+1481.34261551(1+i)^{-4}1056.2471742(1+i)^{-5}-7035.72$$

We evaluate this at the given values:

i	NPV
(i) 0.0962	371.54990
(ii) 0.1201	−0.11348
(iii) 0.1449	−353.01087
(iv) 0.3066	−2055.22362

We see that the true rate of return is (ii) $i = 0.1201$.

2. *An insurance company sells a 5-year endowment insurance policy to a life aged 58, for whom the lifetable below is appropriate.*

x	l_x	d_x
58	*10000.00*	*177.08*
59	*9822.92*	*199.46*
60	*9623.46*	*224.19*
61	*9399.27*	*251.32*
62	*9147.96*	*280.84*

The annual gross premium is \$79,452. Initial expenses are \$2,640 plus 5% of the first premium. The death benefits are \$400,000. Renewal costs are 2% of each subsequent premium. The interest rate is $i = 0.04$. Reserves are calculated on the basis $i = 0.03$, with mortality following the table.

(a) Calculate the reserves.

On the reserve basis, we compute:

Year	$\ddot{a}_{t:\overline{5-t}\rceil}$	$A_{t:\overline{5-t}\rceil}$	Reserve
1	4.53378424775	0.867949	0
2	3.70538350615	0.892077	62430.669669
3	2.84429719151	0.917157	140877.69954
4	1.94492892948	0.943352	222812.306695
5	1	0.970874	308897.6

(b) Calculate the profit signature.

We conduct the following profit test:

Year	Initial Reserves	Premium	Expenses	Interest	Expected Death Benefits	Expected Res. Payments	Pr_t
0			6,612.60				−6,612.60
1	0	79,452	0	3,178.08	7083.2	61325.1473705	14221.7326295
2	62430.669669	79,452	1,589.04	4987.43849007	8122.22842088	138017.097403	−858.257665
3	140877.69954	79,452	1,589.04	7340.8493862	9318.47796946	217621.627767	−858.59681
4	222812.306695	79,452	1,589.04	9798.88760085	10695.2986775	300638.548409	−859.69279
5	308897.6	79,452	1,589.04	12381.4464	400000	0	−857.9936

The profit signature is then calculated as follows:

Year	Pr_t	P(in force)	Π_t
0	−6612.60	1	−6612.60
1	14221.7326295	1	14221.7326295
2	−858.257665	0.982292	−843.059638268
3	−858.59681	0.962346	−826.267205716
4	−859.69279	0.939927	−808.048465026
5	−857.9936	0.914796	−784.889113306

(c) Calculate the profit margin at a risk discount rate of $i = 0.04$.

At risk discount rate $i = 0.04$, the NPV is

$$14221.73263(1.04)^{-1} - 843.0596383(1.04)^{-2} - 826.2672057(1.04)^{-3} - 808.0484650(1.04)^{-4} - 784.8891133(1.04)^{-5} - 6612.60 = \$4,212.2935$$

The EPV of premiums received is

$$79452(1 + 0.982292(1.04)^{-1} + 0.962346(1.04)^{-2} + 0.939927(1.04)^{-3} + 0.914796(1.04)^{-4}) = \$353,705.815689$$

The profit margin is therefore $\frac{4212.2934968}{353705.815689} = 1.1909\%$.

3. *For the policy in Question 2:*

   (a) Calculate the reserves and profit signature for a general premium. [You may assume that P is such that the reserves are zero in Year 1]

   The reserves in given in the following table:

Year	$\ddot{a}_{t:\overline{5-t}\rceil}$	$A_{t:\overline{5-t}\rceil}$	Reserve
1	4.53378424775	0.867949	0
2	3.70538350615	0.892077	$356830.8 - 3.70538350615P$
3	2.84429719151	0.917157	$366862.8 - 2.84429719151P$
4	1.94492892948	0.943352	$377340.8 - 1.94492892948P$
5	1	0.970874	$388349.6 - P$

   The profit test for premium P becomes

Year	Initial Reserves	Premium	Expenses	Interest	Expected Death Benefits	Exp
			$2640 + 0.05P$			
	0	P	0	$0.04P$	7083.2	$350512.040194 - 3.639$
	$356830.8 - 3.70538350615P$	P	$0.02P$	$14273.232 - 0.109015340246P$	8122.22842088	$359413.441348 - 2.786$
	$366862.8 - 2.84429719151P$	P	$0.02P$	$14674.512 - 0.0745718876604P$	9318.47796946	$368550.19517 - 1.899$
	$377340.8 - 1.94492892948P$	P	$0.02P$	$15093.632 - 0.0385971571792P$	10695.2986775	$377966.225762 - 0.9732$
	$388349.6 - P$	P	$0.02P$	$15533.984 - 0.0008P$	400000	

The profit signature is then calculated as follows:

Year	$\Pr_t$	P(in force)	Π_t
0	$-2640 - 0.05P$	1	$-2640 - 0.05P$
1	$4.67976857502P - 357595.240194$	1	$4.67976857502P - 357595.240194$
2	$3568.362231 - 0.04785673361P$	0.982292	$3505.17367261 - 0.0470092865712P$
3	$3668.638861 - 0.03924959314P$	0.962346	$3530.49993333 - 0.0377716889599P$
4	$3772.907561 - 0.030263269441P$	0.939927	$3546.25768509 - 0.0284452640559P$
5	$3883.584 - 0.0208P$	0.914796	$3552.68710886 - 0.0190277568P$

(b) Calculate the premium that gives an internal rate of return of $i = 0.12$.

At risk discount rate $i = 0.12$, the NPV is

$$(4.67976857502P - 357595.240194)(1.12)^{-1} + (3505.17367261 - 0.0470092865712P)(1.12)^{-2} + (3530.49$$
$$= 4.03512980179P - 312344.620404$$

To get an i.r.r. of $i = 0.12$, we need this to equal 0. That is, we need to solve

$$4.03512980179P - 312344.620404 = 0$$
$$P = \frac{312344.620404}{4.03512980179}$$
$$= \$77,406.34$$

4. *For a 5-year term insurance policy with death benefit \$650,000 sold to a life aged 52, with the following lifetable:*

x	l_x	d_x
61	*10000.00*	*129.66*
62	*9870.34*	*148.82*
63	*9721.53*	*170.44*
64	*9551.08*	*194.73*
65	*9356.36*	*221.83*

an actuary performs the following profit test without reserves:

Year	*Premium*	*Expenses*	*Interest*	*Expected Death Benefits*	Pr_t
0		*200*			−200
1	*11300*	*0*	*452*	8427.9	3324.1
2	*11300*	*226*	*442.96*	9800.3716184	1716.5883816
3	*11300*	*226*	*442.96*	11395.9428197	121.0171803
4	*11300*	*226*	*442.96*	13252.3756476	−1735.4156476
5	*11300*	*226*	*442.96*	15410.8542211	−3893.8942211

Calculate the reserves needed to ensure that all cash flows are non-negative.

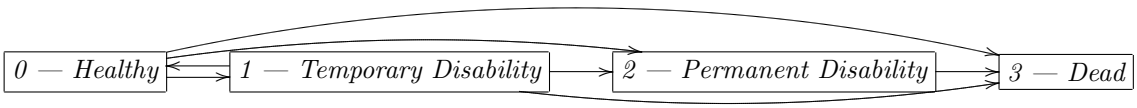
From the interest in the profit test, we see that the interest rate is $i = \frac{452}{11300} = 0.04$
For Year 5, the reserve needed is $3893.8942211(1.04)^{-1} = 3744.12905875$. The expected reserve payment in Year 4 is therefore $\frac{9356.36}{9551.08} \times 3744.12905875 = 3667.79666385$. The expected loss in Year 4 is therefore $1735.4156476 + 3667.79666385 = 5403.21231145$. The reserve in Year 4 is $5403.21231145(1.04)^{-1} = 5195.39645331$. The expected reserve payment in Year 3 is therefore $\frac{9551.08}{9721.53} \times 5195.39645331 = 5104.30427693$. The loss in Year 3 is $5104.30427693 - 121.0171803 = 4983.28709663$. The reserve for Year 3 is therefore $4983.28709663(1.04)^{-1} = 4791.6222083$. The expected reserve payment in Year 2 is therefore $\frac{9721.53}{9870.34} \times 4791.6222083 = 4719.38140395$. Therefore, the expected loss is $4719.38140395 - 1716.5883816 = 3002.79302235$. This means the reserve in Year 2 is $3002.79302235(1.04)^{-1} = 2887.30098303$ and the expected reserve payment in Year 1 is $0.987034 \times 2887.30098303 = 2849.86423848$.

The new table with reserves is

Year	Initial Reserves	Premium	Expenses	Interest	Expected Death Benefits	Expected Reserve Payments	Pr_t
0	0		200			−200	
1	0	11300	0	452	8427.9	2849.86423848	474.23576152
2	2887.30098303	11300	226	558.45203932	9800.3716184	4719.38140395	0
3	4791.6222083	11300	226	634.624888332	11395.9428197	5104.30427693	0
4	5195.39645331	11300	226	650.775858132	13252.3756476	3667.79666385	0
5	3744.12905875	11300	226	592.725162352	15410.8542211	0	0

Standard Questions

5. *An insurer sells a 5-year disability income protection policy for a life aged 55. The policy has the following state diagram:*



The transition probabilities are given in the following table:

x	p_x^{01}	p_x^{02}	p_x^{03}	p_x^{10}	p_x^{12}	p_x^{13}	p_x^{23}
55	*0.05678216*	*0.01488069*	*0.009911805*	*0.01639633*	*0.08036063*	*0.04682560*	*0.1822381*
56	*0.09057920*	*0.02020286*	*0.011906520*	*0.01675757*	*0.12722420*	*0.06748682*	*0.2543372*
57	*0.13774081*	*0.02382313*	*0.016797271*	*0.01624059*	*0.17690254*	*0.12421022*	*0.4026589*
58	*0.18931439*	*0.03918915*	*0.023217273*	*0.01803712*	*0.28036575*	*0.14488319*	*0.6746820*
59	*0.22145329*	*0.04847626*	*0.028821625*	*0.01593516*	*0.35974325*	*0.19971552*	*0.8580170*

The probability of being in each state at the end of each year is

t	${}_tp_{55}^{00}$	${}_tp_{55}^{01}$	${}_tp_{55}^{02}$	${}_tp_{55}^{03}$
1	*0.91842535*	*0.05678216*	*0.01488069*	*0.009911805*
2	*0.80669658*	*0.1279647*	*0.03538679*	*0.02995188*
3	*0.66489142*	*0.1984697*	*0.05945463*	*0.0771842*
4	*0.50110424*	*0.2363644*	*0.08915138*	*0.1733800*
5	*0.35516526*	*0.2113329*	*0.07740444*	*0.3560974*

The policy pays a benefit of \$79,000 at the end of any year if the life is disabled at that time (State 1 or State 2), and pays a death benefit of \$734,000 at the end of the year when the life dies (enters State 3). The interest rate is $i = 0.06$. *Initial expenses are \$800 plus 25% of the first premium. Renewal expenses are 3% of each subsequent premium. The premium is \$126,060 at the start of each year. Use a profit test to calculate the reserves for each year in each state using a reserve rate of* $i = 0.04$ *and calculate the profit margin at a risk discount rate of* $i = 0.12$.

First we perform a profit test without reserves at the reserve rate $i = 0.04$ in the healthy state:

Year	Premium	Expenses	Interest	Expected Dis Benefits	Expected Death Benefits	Pr_t
0		32315			−32315	
1	126060	0	5042.4	5661.36515	7275.26487	118165.76998
2	126060	3781.80	4891.128	8751.78274	8739.38568	109678.15958
3	126060	3781.80	4891.128	12763.55126	12329.196914	102076.579826
4	126060	3781.80	4891.128	18051.77966	17041.478382	92076.069958
5	126060	3781.80	4891.128	21324.43445	21155.07275	84689.8208

in the temporary disability state:

Year	Premium	Expenses	Interest	Expected Dis Benefits	Expected Death Benefits	Pr_t
2	0	0	0	72344.69319	49535.32588	−121880.01907
3	0	0	0	67904.38601	91170.30148	−159074.68749
4	0	0	0	66129.29551	106344.26146	−172473.55697
5	0	0	0	61963.59628	146591.19168	−208554.78796

and in the permanent disability state:

Year	Premium	Expenses	Interest	Expected Dis Benefits	Expected Death Benefits	$\Pr_t$
2	0	0	0	58907.3612	186683.5048	−245590.866
3	0	0	0	47189.9469	295551.6326	−342741.5795
4	0	0	0	25700.122	495216.588	−520916.71
5	0	0	0	11216.657	629784.478	−641001.135

We compute reserve payments in state 2. Since it is impossible to get from State 2 to State 0 or State 1, we can calculate these reserves without considering State 0 or State 1.

	Permanent Disability		
Year	Loss without Reserves	Expected Reserve Payment	Reserve
2	245590.866	$727984.820744 \times (1 - 0.2543372) = 542831.199793$	758098.140186
3	342741.5795	$693678.426068 \times (1 - 0.4026589) = 414362.634074$	727984.820744
4	520916.71	$616347.245192 \times (1 - 0.6746820) = 200508.853111$	693678.426068
5	641001.135	0	616347.245192

Now we compute the reserve payments in States 0 and 1:

	0 — Healthy				1 — Temporary Disability		
Year	Loss without Reserves	Exp. Reserve Payment (State 1)	Exp. Reserve Payment (State 2)	Reserve	Loss without Reserves	Exp. Reserve Payment (State 1)	Exp. Reserve Payment (State 2)
1	−118165.76998	32038.3784664	11281.0234137	0			
2	−109678.15958	48397.0556352	14707.3754156	0	121880.01907	421317.460519	43605.0232586
3	−102076.579826	61430.8235933	16525.5913224	0	159074.68749	304452.586947	92151.4891274
4	−92076.069958	37963.8677542	24154.1246439	0	172473.55697	111639.76703	179714.763965
5	−84689.8208	0	0	0	208554.78796	0	0

With these reserves, we adjust the profit tests, also using the interest rate $i = 0.04$. In the healthy state:

Year	Premium	Expenses	Interest	Expected Death Benefits	Expected Dis Benefits	Expected $\Pr_t$ Reserve Payments	
0		32315			−32315		
1	126060	0	5042.4	5661.36515	7275.26487	43319.4018801	74846.3680999
2	126060	3781.80	4891.128	8751.78274	8739.38568	63104.4310508	46573.7285292
3	126060	3781.80	4891.128	12763.55126	12329.196914	77956.4149157	24120.1649103
4	126060	3781.80	4891.128	18051.77966	17041.478382	62117.9923981	29958.0775599
5	126060	3781.80	4891.128	21324.43445	21155.07275		84689.8208

in the temporary disability state:

Year	Reserves	Premium	Expenses	Interest	Expected Death Benefits	Expected Dis Benefits	Expected Reserve Payments	$\Pr_t$
2	121880.01907	0	0	4875.2007628	72344.69319	49535.32588	464922.483778	0
3	159074.68749	0	0	6362.9874996	67904.38601	91170.30148	396604.076074	0
4	172473.55697	0	0	6898.9422788	66129.29551	106344.26146	291354.530995	0
5	208554.78796	0	0	8342.1915184	61963.59628	146591.19168		0

and in the permanent disability state:

Year	Reserves	Premium	Expenses	Interest Benefits	Expected Death Benefits	Expected Dis Payments	Expected Reserve	$\Pr_t$
2	758098.140186	0	0	30323.9256074	58907.3612	186683.5048	542831.199793	0
3	727984.820744	0	0	29119.3928298	47189.9469	295551.6326	414362.634074	0
4	693678.426068	0	0	27747.1370427	25700.122	495216.588	200508.853111	0
5	616347.245192	0	0	24653.8898077	11216.657	629784.478	0	0

Since the profit vector is zero for both disabled states, we only need to consider the healthy state when computing the profit signature. The profit signature is then calculated as follows:

Year	$\Pr_t$	P(healthy) Π_t	
0	−323151188	1	−32315
1	74846.3680999	1	74846.3680999
2	46573.7285292	0.91842535	42774.4929252
3	24120.1649103	0.80669658	19457.6545422
4	29958.0775599	0.66489142	19918.8687293
5	84689.8208	0.50110424	42438.4282877

At risk discount rate $i = 0.10$, the NPV is

$$74846.3680999\times1.10^{-1}+42774.4929252\times1.10^{-2}+19457.6545422\times1.10^{-3}+19918.8687293\times1.10^{-4}+42438.4282877\times1.10^{-5}-32315 = 12565$$

The EPV of premiums is

$$126060\left(1+0.91842535(1.10)^{-1}+0.80669658(1.10)^{-2}+0.66489142(1.10)^{-3}+0.50110424(1.10)^{-4}\right) = 421472.436373$$

Therefore the profit margin is $\frac{125652.577665}{421472.436373} = 29.81\%$.