We are dealing with the situation where X follows a mixture of Poisson distributions, and where we want to estimate the moments of $\log(\Lambda)$, where Λ is the latent Poisson mean. Since we only have one observation of X for each Λ, we want to obtain an estimator for $\log(\Lambda)$ from this single observation X. One approach is the posterior mean estimator. We assume an improper uniform prior on the interval $[a, \infty]$. The difficulty here is that if we observe the value $X = 0$, the posterior distribution will be heavily weighted towards $\log(\Lambda) \approx -\infty$, which may lead to bad results. However, from a practical point of view, it is impossible to distinguish results for very small values of $\log(\Lambda)$, since they are all almost certain to result in $X = 0$. We therefore, for convenience choose the lower bound a to represent the smallest value that we are able to detect (more details on the choice of a will follow in Section 2.

1 Estimating the Posterior Mean

Let $U = \log(\Lambda)$. Suppose we have chosen the lower bound a. Now the likelihood of u from X is proportional to $e^{-\lambda}\lambda^X$, where $\lambda = e^u$. The posterior mean is therefore

$$\frac{\int_a^\infty u e^{-e^u} e^{ux}\, du}{\int_a^\infty e^{-e^u} e^{ux}\, du}$$

we can perform a change of variable in the denominator:

$$\int_a^\infty e^{-e^u} e^{ux}\, du = \int_{e^a}^\infty e^{-\lambda}\lambda^{x-1}\, d\lambda = \Gamma(x; e^a)$$

an incomplete gamma function. Recall that by integrating by parts, we have

$$\Gamma(x; l) = (x-1)! e^{-l} \left(1 + l + \frac{l^2}{2} + \cdots + \frac{l^{x-1}}{(x-1)!}\right)$$

For the numerator, we can use integration by parts: if we let

$$g(x, a) = \int_a^\infty u e^{-e^u} e^{ux}\, du$$

then we have

$$
\begin{aligned}
g(x,a) &= \int_a^\infty \left(ue^{-e^u}\right) e^{ux}\,du \\
&= \left[\frac{ue^{-e^u}e^{ux}}{x}\right]_a^\infty - \frac{1}{x}\int_a^\infty e^{ux}\left(e^{-e^u} - ue^u e^{-e^u}\right)\,du \\
&= -\frac{ae^{-e^a}e^{ax}}{x} - \frac{1}{x}\int_a^\infty e^{ux}e^{-e^u}\,du + \frac{1}{x}\int_a^\infty ue^{u(x+1)}e^{-e^u}\,du \\
&= -\frac{ae^{-e^a}e^{ax}}{x} - \frac{\Gamma(x;e^a)}{x} + \frac{g(x+1,a)}{x} \\
g(x+1,a) &= xg(x,a) + \Gamma(x,e^a) + ae^{-e^a}e^{ax} \\
&= x!g(1,a) + \sum_{m=1}^{x}\frac{x!}{m!}\left(\Gamma(m,e^a) + ae^{-e^a}e^{am}\right) \\
&= x!g(1,a) + e^{-e^a}\sum_{m=1}^{x}\frac{x!}{m!}\left(ae^{am} + (m-1)!\sum_{k=0}^{m-1}\frac{e^{ka}}{k!}\right) \\
&= x!g(1,a) + e^{-e^a}\left(a\sum_{m=1}^{x}\frac{x!e^{am}}{m!} + \sum_{m=1}^{x}\sum_{k=0}^{m-1}\frac{x!e^{ka}}{k!m}\right) \\
&= x!g(1,a) + x!e^{-e^a}\left(a\sum_{m=1}^{x}\frac{e^{am}}{m!} + \sum_{k=0}^{x}\frac{e^{ka}}{k!}\sum_{m=k+1}^{x}\frac{1}{m}\right)
\end{aligned}
$$

If we let $h(x,a) = \frac{g(x,a)}{\Gamma(x,e^a)}$, then the recurrence becomes

$$h(x+1,a) = (h(x,a)x+1)\frac{\Gamma(x,e^a)}{\Gamma(x+1,e^a)} + \frac{ae^{-e^a}e^{ax}}{\Gamma(x+1,e^a)}$$

From the formula, we have

$$\Gamma(x+1,e^a) = x\Gamma(x,e^a) + e^{-e^a}e^{xa}$$

giving us the recurrence

$$h(x+1,a) = h(x,a) + \frac{1}{x} + \frac{\left(a - h(x,a) - \frac{1}{x}\right)e^{-e^a}e^{ax}}{\Gamma(x+1,e^a)}$$

To start this recurrence, we need to calculate $h(0,a)$, which is based on $g(0,a) = \int_a^\infty ue^{-e^u}\,du$. If we let $g(0,0) = c$, then we have $g(0,a) = c +$

$\int_a^0 ue^{-e^u}\,du$. We can now expand a Taylor series to get

$$\begin{aligned}
\int_a^0 ue^{-e^u}\,du &= \int_a^0 u\sum_{n=0}^{\infty}\frac{(-e^u)^n}{n!}\,du \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\int_a^0 ue^{nu}\,du \\
&= -\frac{a^2}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\left[u\frac{e^{nu}}{n}\right]_a^0-\int_a^0\frac{e^{nu}}{n}\,du\right) \\
&= -\frac{a^2}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(-a\frac{e^{na}}{n}-\left[\frac{e^{nu}}{n^2}\right]_a^0\right) \\
&= -\frac{a^2}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(-a\frac{e^{na}}{n}-\frac{1-e^{na}}{n^2}\right) \\
&= -\frac{a^2}{2}+a\sum_{n=0}^{\infty}\frac{(-1)^{n+1}e^{na}}{n!n}+\sum_{n=0}^{\infty}\frac{(-1)^n e^{na}}{n!n^2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!n^2}
\end{aligned}$$

Now we have

$$\begin{aligned}
\frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{x^n}{n!n}\right) &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!} \\
&= \frac{e^x-1}{x} \\
\sum_{n=1}^{\infty}\frac{x^n}{n!n} &= \int_0^x\frac{e^t-1}{t}\,dt
\end{aligned}$$

and

$$
\begin{aligned}
\frac{d}{dx}\left(\sum_{n=1}^{\infty}\frac{x^n}{n!n^2}\right) &= \sum_{n=1}^{\infty}\frac{x^{n-1}}{n!n} \\
&= 1+\frac{1}{x}\int_0^x \frac{e^t-1}{t}\,dt \\
\sum_{n=1}^{\infty}\frac{x^n}{n!n^2} &= \int_0^x 1+\frac{1}{s}\int_0^s \frac{e^t-1}{t}\,dt\,ds \\
&= x+\int_0^x\int_0^s \frac{e^t-1}{st}\,dt\,ds \\
&= x+\int_0^x\int_t^x \frac{e^t-1}{st}\,ds\,dt \\
&= x+\int_0^x \frac{e^t-1}{t}\left(\log(x)-\log(t)\right)\,dt
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\int_a^0 ue^{-e^u}\,du &= a\sum_{n=0}^{\infty}\frac{(-1)^{n+1}e^{na}}{n!n}+\sum_{n=0}^{\infty}\frac{(-1)^{n}e^{na}}{n!n^2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!n^2} \\
&= \int_{-e^a}^0 \frac{e^t-1}{t}\left(a+\log(-t)-\log(e^a)\right)\,dt+\int_{-1}^0 \frac{e^t-1}{t}\left(\log(-t)-\log(1)\right)\,dt \\
&= \int_{-e^a}^0 \frac{e^t-1}{t}\log(-t)\,dt+\int_{-1}^0 \frac{e^t-1}{t}\log(-t)\,dt \\
\int_a^0 ue^{-e^u}\,du &= \int_{e^a}^1 \frac{e^{-t}\log(t)}{t}\,dt
\end{aligned}
$$

$$
\begin{aligned}
g(x,a) &= \int_a^\infty u e^{-e^u} e^{ux}\,du \\
&= \int_a^0 u e^{-e^u} e^{ux}\,du + g(x,0) \\
&= \int_a^0 u \sum_{n=0}^\infty \frac{(-1)^n e^{nu} e^{ux}}{n!}\,du + g(x,0) \\
&= g(x,0) + \sum_{n=0}^\infty \frac{(-1)^n}{n!} \int_a^0 u e^{(x+n)u}\,du \\
&= g(x,0) + \sum_{n=0}^\infty \frac{(-1)^n}{n!} \left(\left[\frac{u e^{(x+n)u}}{x+n} \right]_a^0 - \int_a^0 \frac{e^{(x+n)u}}{x+n}\,du \right) \\
&= g(x,0) + \sum_{n=0}^\infty \frac{(-1)^n e^{(x+n)a}}{n!} \left(-\frac{a}{x+n} + \frac{1}{(x+n)^2} \right) - \sum_{n=0}^\infty \frac{(-1)^n}{n!(x+n)^2}
\end{aligned}
$$

$$
\begin{aligned}
g(x+1,0) = x!g(1,0) + x!e^{-1} \left(\sum_{k=0}^x \frac{e^{ka}}{k!} \sum_{m=k+1}^x \frac{1}{m} \right) \\
= x!g(1,0) + e^{-1} \left(\sum_{k=0}^x \sum_{m=k+1}^x \prod_{l \geqslant k, l \neq m} l \right) \\
= x!g(1,0) + e^{-1} \left(\sum_{m=1}^x \sum_{k=0}^{m-1} \prod_{l \neq m} l \right) \\
= x!g(1,0) + e^{-1} \left(\sum_{m=1}^x m \prod_{l \neq m} l \right) \\
= x!g(1,0) + e^{-1} \left(\sum_{m=1}^x \prod_{k \neq m} k + x!x \right)
\end{aligned}
$$

$$
g(x+1,0) = xg(x,0) + \Gamma(x,1)
$$

We also have the recurrence

$$
\Gamma(x,1) = (x-1)!e^{-1} \sum_{m=0}^{x-1} \frac{1}{m!} = (x-1)\Gamma(x-1,1) + e^{-1}
$$

Numerically, we get $g(1,0) = 0.2193839$.

We have recurrences for the numerator and denominator, but computationally, both increase to infinity for moderate n, so for computational purposes, we need to derive a recurrence for the ratio.

We have that the numerator satisfies the recurrence

$$p_{n+1} = np_n + ae^{an}e^{-e^a} + \Gamma(n, e^a)$$

while the denominator satisfies

$$\Gamma(n+1, e^a) = n\Gamma(n, e^a) + e^{an}e^{-e^a}$$

This means the ratio is

$$\begin{aligned} r_{n+1} = \frac{p_{n+1}}{\Gamma(n+1, e^a)} &= \frac{np_n + ae^{an}e^{-e^a} + \Gamma(n, e^a)}{n\Gamma(n, e^a) + e^{an}e^{-e^a}} \\ &= r_n \frac{n\Gamma(n, e^a)}{n\Gamma(n, e^a) + e^{an}e^{-e^a}} + \frac{\Gamma(n, e^a) + ae^{an}e^{-e^a}}{n\Gamma(n, e^a) + e^{an}e^{-e^a}} \end{aligned}$$

If we define $t_n = \frac{e^{na}e^{-e^a}}{\Gamma(n+1, e^a)}$, then our recurrence becomes

$$r_{n+1} = r_n(1 - t_n) + \frac{1}{n} + \left(a - \frac{1}{n}\right) t_n$$

Substituting $x = e^a$, we get

$$t_n = \frac{x^n e^{-x}}{e^{-x} n! \sum_{i=0}^{n} \frac{x^i}{i!}} = \frac{\frac{x^n}{n!}}{\sum_{i=0}^{n} \frac{x^i}{i!}}$$

Now we have

$$(1 - t_n) \sum_{i=0}^{n} \frac{x^i}{i!} = \sum_{i=0}^{n-1} \frac{x^i}{i!}$$

which gives

$$\begin{aligned} t_n &= \frac{x}{n} t_{n-1}(1 - t_n) \\ t_n \left(1 + \frac{x}{n} t_{n-1}\right) &= \frac{x}{n} t_{n-1} \\ t_n &= \frac{x t_{n-1}}{x t_{n-1} + n} \end{aligned}$$

We see that as $a \to -\infty$, we will get $at_n \to 0$ and $t_n \to 0$, so that our recurrencurrence becomes $t_{n+1} = t_n + \frac{1}{n}$. This means that t_n is not sensitive to the choice of a for $n \geqslant 1$ and a close to $-\infty$. We can simplify calculations by choosing this limiting value for r_n when $n \geqslant 1$.

2 Choosing the Lower Bound

The choice of lower bound a has a large impact on the posterior mean for an observation of zero, but a small impact on the posterior mean for other observations. One way to choose a is to cut off values of λ which will not be distinguishable in the dataset. For example, if our dataset has size n, and the parameter is $\lambda = e^u$, the probability of observing a zero value is e^{-e^u}, so the probability of observing all zero values is e^{-ne^u}. This means that with 95% confidence, if u is smaller than the solution to $e^{-ne^u} = 0.95$, then the dataset will contain all zeros. We therefore choose to set a to this solution

$$a = \log\left(-\frac{\log(0.95)}{n}\right) = -2.970195 - \log(n)$$

Recall that our objective is to estimate the underlying variance of $\log(\Lambda)$, which we plan to estimate as $\mathrm{Var}(R) - \mathbb{E}\left(\mathrm{Var}(R|\Lambda)\right)$. When we set r_0, we are controlling the second term, while creating a bias in our estimator R. Suppose that we compare our lower bound a with a smaller lower bound a'. The bias from using a instead of a' is $a - a'$. The variance $\mathrm{Var}(R|\Lambda = \lambda)$ is $P(X > 0)\,\mathrm{Var}(R|\Lambda = \lambda, X > 0) + P(X = 0)P(X > 0)\left(\mathbb{E}(R|X > 0) - r_0\right)^2$. As always, this variance will be averaged over all samples, so that the best choice of r_0 is the one such that

$$\begin{aligned}
\frac{d}{da}\mathbb{E}\left((a-\lambda)_+^2\right) &= \frac{1}{n}\frac{d}{da}\mathbb{E}\left(P(X=0)P(X>0)\left(\mathbb{E}(R|X>0) - r_0\right)^2\right) \\
\mathbb{E}\left((a-\lambda)_+\right) &= \frac{1}{n}\left(\mathbb{E}\left(P(X=0)P(X>0)\left(r_0 - \mathbb{E}(R|X>0)\right)\right)\right)\frac{dr_0}{da} \\
&= \frac{1}{n}\left(r_0 P(X=0)P(X>0) - \mathbb{E}\left(P(X=0)P(X>0)\mathbb{E}(R|X>0)\right)\right)\frac{dr_0}{da} \\
&= P(\Lambda < a)\mathbb{E}_{\Lambda<a}\left(P(X=0)P(X>0)\left(r_0 - \mathbb{E}(R|X>0)\right)\right) \\
&\qquad + P(\Lambda > a)\mathbb{E}_{\Lambda>a}\left(P(X=0)P(X>0)\left(r_0 - \mathbb{E}(R|X>0)\right)\right)
\end{aligned}$$

When $\Lambda < e^a$, we have that $P(X=0) = e^{-\Lambda} \approx 1 - \Lambda$, $P(X > 0) \approx \Lambda - \frac{\Lambda^2}{2}$, and

If we assume $\mathbb{E}(R|X > 0) \approx r_1$ then our expression becomes:

$$\mathbb{E}\left((a-\lambda)_+\right) \approx \frac{1}{n}(r_0 - r_1)\mathbb{E}\left(P(X=0)P(X>0)\right)\frac{dr_0}{da}$$

Recall that

$$
\begin{aligned}
r_0 &= \frac{\int_a^\infty ue^{-e^u}\,du}{\int_a^\infty e^{-e^u}\,du} \\
\frac{dr_0}{da} &= \frac{d}{da}\left(\frac{\int_a^\infty ue^{-e^u}\,du}{\int_a^\infty e^{-e^u}\,du}\right) \\
&= \frac{ae^{-e^a}\int_a^\infty e^{-e^u}\,du - e^{-e^a}\int_a^\infty ue^{-e^u}\,du}{\left(\int_a^\infty e^{-e^u}\,du\right)^2} \\
&= \frac{e^{-e^a}}{\int_a^\infty e^{-e^u}\,du}(a - r_0)
\end{aligned}
$$

For very negative u, we have $e^{-e^u} \approx 1$, giving

$$
\frac{dr_0}{da} \approx \frac{a - r_0}{c - a}
$$

which has solution $r_0 \approx \frac{a}{2}$.

$$
\mathbb{E}\left((a-\lambda)_+\right) \approx \frac{r_0 - r_1}{2n}\mathbb{E}\left(e^{-\Lambda} - e^{-2\Lambda}\right)
$$

3 Conditional Variance

Having determined that our posterior mean estimate given an observation $X = n$ is r_n, we now need to find the conditional mean of r_X when $X \sim \mathrm{Po}(\lambda)$. This is given by

$$
m = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}r_n
$$

The mean of $(r_X)^2$ is given by

$$
s = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(r_n)^2
$$

The variance of r_X is therefore

$$
s - m^2 = e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}(r_n)^2 - e^{-2\lambda}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^{n+m}}{n!m!}r_m r_n
$$

Recall that our recurrence is

$$r_{n+1} = r_n(1-t_n) + \frac{1}{n} + \left(a - \frac{1}{n}\right) t_n$$

where

$$t_n = \frac{e^{-a} t_{n-1}}{e^{-a} t_{n-1} + n}$$

We expand this recurrence to get

$$\begin{aligned}
r_{n+1} &= r_n(1-t_n) + \frac{1}{n} + \left(a - \frac{1}{n}\right) t_n \\
&= \left(r_{n-1}(1-t_{n-1}) + \frac{1}{n-1} + \left(a - \frac{1}{n-1}\right) t_{n-1}\right)(1-t_n) + \frac{1}{n} + \left(a - \frac{1}{n}\right) t_n \\
&= r_{n-1}(1-t_{n-1})(1-t_n) + \frac{1}{n-1}(1-t_{n-1})(1-t_n) + \frac{1}{n}(1-t_n) + a(t_n + t_{n-1}(1-t_n)) \\
&= r_{n-2}s_{n-2}s_{n-1}s_n + \frac{1}{n-2}s_{n-2}s_{n-1}s_n + \frac{1}{n-1}s_{n-1}s_n + \frac{1}{n}s_n + a(t_n + t_{n-1}s_n + t_{n-2}s_{n-1}s_n) \\
&= \cdots \\
&= r_1 \prod_{i=1}^{n} s_i + \sum_{i=1}^{n} \frac{1}{i} \prod_{j=i}^{n} s_j + a \sum_{i=1}^{n} t_i \prod_{j=i+1}^{n} s_j
\end{aligned}$$

where

$$s_n = 1 - t_n = \frac{\sum_{i=0}^{n-1} \frac{e^{ia}}{i!}}{\sum_{i=0}^{n} \frac{e^{ia}}{i!}}$$

This gives

$$\prod_{j=i+1}^{n} s_j = \frac{\sum_{k=0}^{i} \frac{e^{ka}}{k!}}{\sum_{k=0}^{n} \frac{e^{ka}}{k!}}$$

and

$$t_i \prod_{j=i+1}^{n} s_j = \frac{\frac{e^{ia}}{i!}}{\sum_{k=0}^{n} \frac{e^{ka}}{k!}}$$

so

$$\sum_{i=1}^{n} t_i \prod_{j=i+1}^{n} s_j = 1 - \frac{1}{\sum_{k=0}^{n} \frac{e^{ka}}{k!}}$$

$$\sum_{i=1}^{n} \frac{1}{i} \prod_{j=i}^{n} s_j = \frac{\sum_{i=1}^{n} \sum_{k=0}^{i-1} \frac{e^{ka}}{k!i}}{\sum_{k=0}^{n} \frac{e^{ka}}{k!}} = \frac{\sum_{k=0}^{n-1} \sum_{i=k+1}^{n} \frac{e^{ka}}{k!i}}{\sum_{k=0}^{n} \frac{e^{ka}}{k!}}$$

If we write $\tau_n = \prod_{j=0}^{n-1} s_j = \frac{1}{\sum_{k=0}^{n-1} \frac{e^{ka}}{k!}}$ then we get

$$r_n = r_1\tau_n + \tau_n \sum_{i=1}^{n-1} \frac{1}{i} \sum_{k=0}^{i-1} \frac{e^{ka}}{k!} + a\left(1 - \tau_n\right)$$

If we let $\nu_n = \sum_{i=1}^{n-1} \frac{1}{i} \sum_{k=0}^{i-1} \frac{e^{ka}}{k!}$, then we have

$$\begin{aligned}
\nu_n &= \sum_{i=1}^{n-1} \frac{1}{i} \sum_{k=0}^{i-1} \frac{e^{ka}}{k!} \\
&= \sum_{i=1}^{n-1} \sum_{k=0}^{i-1} \frac{e^{ka}}{(k+1)!} - \frac{e^{ka}}{k!} \left(\frac{1}{k+1} - \frac{1}{i} \right) \\
&= \sum_{i=1}^{n-1} \sum_{k=0}^{i-1} \frac{e^{ka}}{(k+1)!} - \sum_{i=1}^{n-1} \sum_{k=0}^{i-1} \frac{(i-1-k)e^{ka}}{(k+1)!i} \\
\nu_{n+1} &= \nu_n + \frac{1}{n} \sum_{k=0}^{n-1} \frac{e^{ka}}{k!} \\
&= \nu_n + \frac{1}{n\tau_n} \\
\nu_{n+1}\tau_{n+1} &= \tau_{n+1}\nu_n + \frac{\tau_{n+1}}{n\tau_n} \\
&= s_n \left(\tau_n \nu_n + \frac{1}{n} \right) \\
&= s_n \left(s_{n-1} \left(\tau_{n-1}\nu_{n-1} + \frac{1}{n-1} \right) + \frac{1}{n} \right)
\end{aligned}$$

and

$$r_n = r_1\tau_n + \tau_n\nu_n + a\,(1-\tau_n)$$

This gives

$$\begin{aligned}
r_k r_l &= (r_1\tau_k + \tau_k\nu_k + a\,(1-\tau_k))(r_1\tau_l + \tau_l\nu_l + a\,(1-\tau_l)) \\
&= \tau_k\tau_l(r_1\nu_k - a)(r_1\nu_l - a) + \tau_k a(r_1\nu_l - a) + \tau_l a(r_1\nu_k - a) + a^2
\end{aligned}$$

In practice, we will need to estimate the conditional variance from an observation of X. Given the observation $X = n$, recall that our posterior density of Λ is $\pi_{\Lambda|X}(\lambda|n) = \frac{e^{-\lambda}\lambda^{n-1}}{\int_{e^a}^{\infty} e^{-\lambda}\lambda^{n-1}}$ Therefore, the posterior mean of conditional variance is

$$\int_{e^a}^{\infty} \frac{e^{-\lambda}\lambda^{n-1}}{\int_{e^a}^{\infty} e^{-\lambda}\lambda^{n-1}\,d\lambda} \left(e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^m}{m!}(r_m)^2 - e^{-2\lambda} \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{\lambda^{m+k}}{m!k!} r_m r_k \right) d\lambda$$

$$= \frac{1}{\int_{e^a}^{\infty} e^{-\lambda}\lambda^{n-1}\,d\lambda} \left(\sum_{m=0}^{\infty} \frac{(r_m)^2}{m!} \int_{e^a}^{\infty} e^{-2\lambda}\lambda^{n+m-1}\,d\lambda - \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{r_m r_k}{m!k!} \int_{e^a}^{\infty} e^{-3\lambda}\lambda^{n+m+k-1}\,d\lambda \right)$$

$$= \sum_{m=0}^{\infty} \frac{(r_m)^2}{2^{n+m}m!} \frac{\Gamma\left(n+m; \frac{e^a}{2}\right)}{\Gamma(n; e^a)} - \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} \frac{r_m r_k}{3^{n+m+k} m!k!} \frac{\Gamma\left(n+m+k; \frac{e^a}{3}\right)}{\Gamma(n; e^a)}$$

$$= \sum_{m=0}^{\infty} \left(\frac{(r_m)^2}{2^{n+m}m!} \frac{\Gamma\left(n+m; \frac{e^a}{2}\right)}{\Gamma(n; e^a)} - \left(\sum_{k+l=m}^{\infty} \frac{r_k r_l}{3^{n+m} k!l!} \right) \frac{\Gamma\left(n+m; \frac{e^a}{3}\right)}{\Gamma(n; e^a)} \right)$$

Now recall that

$$\Gamma(x+1; l) = x\Gamma(x; l) + l^x e^{-l}$$

so

$$\begin{aligned}
\frac{\Gamma\left(n+1+m; \frac{e^a}{2}\right)}{\Gamma\left(n+1; e^a\right)} &= \frac{(n+m)\Gamma\left(n+m; \frac{e^a}{2}\right) + 2^{-(n+m)} e^{(n+m)a} e^{-\frac{e^a}{2}}}{n\Gamma\left(n; e^a\right) + e^{na} e^{-e^a}} \\
&= \frac{(n+m)\Gamma\left(n+m; \frac{e^a}{2}\right)}{n\Gamma\left(n; e^a\right)} (1-t_n) + 2^{-(n+m)} e^{ma} e^{\frac{e^a}{2}} t_n
\end{aligned}$$

and

$$\frac{\Gamma\left(n+1+m; \frac{e^a}{3}\right)}{\Gamma\left(n+1; e^a\right)} = \frac{(n+m)\Gamma\left(n+m; \frac{e^a}{3}\right)}{n\Gamma\left(n; e^a\right)} (1-t_n) + 3^{-(n+m)} e^{ma} e^{\frac{2e^a}{3}} t_n$$

We are assuming that a is small enough that $e^{\frac{a}{3}} \approx 0$, so our expression for expected conditional variance becomes

$$\sum_{m=0}^{\infty} \left(\frac{(r_m)^2}{2^{n+m}} \binom{n+m-1}{m} - \left(\sum_{k+l=m}^{\infty} \frac{r_k r_l}{3^{n+m}} \right) \binom{n+m-1}{n-1, k, l} \right)$$

$$= \sum_{m=0}^{\infty} \binom{n+m-1}{m} \left(\frac{(r_m)^2}{2^{n+m}} - \left(\sum_{k+l=m}^{\infty} \binom{m}{k} \frac{r_k r_l}{3^{n+m}} \right) \right)$$

$$= \sum_{m=0}^{\infty} \frac{1}{2^{n+m}} \binom{n+m-1}{m} (r_m)^2 - \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right)^m \binom{n+m-1}{m} \sum_{k+l=m}^{\infty} \frac{1}{2^m} \binom{m}{k} r_k r_l$$

We see that the first term in the first sum is proportional to the probability mass function of a negative binomial distribution with $r = n$ and $\beta = 1$. This has mean n and variance $2n$. In the second sum, it is proportional to the probability mass function of a negative binomial distribution with $r = n$ and $\beta = 2$. This has mean n and variance $6n$. For large n, we can use a normal approximation to this, and see that the probability is almost entirely in the interval $n \pm \sqrt{300n}$ (5 standard deviations either side of the mean). Since $r_m \approx \sum_{k=1}^{m} \frac{1}{k}$ does not change much, we can let this be the range over which we evaluate the expectation. Since r_m can does grow slowly, we will actually use the range $n \pm 20\sqrt{n}$.

Finally, the inner sum in the second term is the expectation over a binomial distribution.

Recalling that $r_m \approx \sum_{i=1}^{m-1} \frac{1}{i}$ for $m > 0$, we have that $(r_m)^2 = \sum_{i,j} \frac{1}{ij}$.

To compute this sum in practice, we need to determine which terms should be included in the sum.