Suppose X follows a mixture of Poisson distributions with mean Λ where Λ has density function $\pi(\lambda)$.

1 Moments of Λ

Suppose we know that $P(X = n) = p_n$. We therefore get the equations

$$\int_0^\infty e^{-\lambda} \frac{\lambda^n}{n!} \pi(\lambda)\, d\lambda = p_n$$

If we replace $e^{-\lambda}$ by its Taylor series, this becomes

$$\begin{aligned}
\int_0^\infty \left(1 - \lambda + \frac{\lambda^2}{2!} - \cdots\right) \frac{\lambda^n}{n!} \pi(\lambda)\, d\lambda &= p_n \\
\int_0^\infty \sum_{k=0}^\infty (-1)^k \frac{\lambda^{n+k}}{k!n!} \pi(\lambda)\, d\lambda &= p_n \\
\sum_{k=0}^\infty \int_0^\infty \frac{(-1)^k \lambda^{n+k}}{k!n!} \pi(\lambda)\, d\lambda &= p_n \\
\sum_{k=0}^\infty \frac{(-1)^k}{k!n!} \mathbb{E}\left(\Lambda^{n+k}\right) &= p_n
\end{aligned}$$

We can now invert this to get the raw moments of Λ. If we write the equations in terms of infinite dimensional matrices, they become

$$F\mu = p$$

where

$$F = \begin{pmatrix} 1 & -1 & \frac{1}{2} & -\frac{1}{6} & \frac{1}{24} & \cdots \\ 0 & 1 & -1 & \frac{1}{2} & -\frac{1}{6} & \cdots \\ 0 & 0 & 1 & -1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

We need the inverse of F. It is easy to see that

$$F^{-1} = \begin{pmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{24} & \cdots \\ 0 & 1 & 1 & \frac{1}{2} & \frac{1}{6} & \cdots \\ 0 & 0 & 1 & 1 & \frac{1}{2} & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This means we can estimate the raw second moment of Λ as

$$\mu_2 = 2p_2 + 3!p_3 + \frac{4!p_4}{2} + \cdots = \sum_{n=2}^{\infty} n(n-1)p_n = \mathbb{E}\left(X(X-1)\right)$$

and the mean of Λ as

$$\mu = p_1 + 2p_2 + \frac{3!p_3}{2} + \frac{4!p_4}{6} + \cdots = \sum_{n=1}^{\infty} np_n = \mathbb{E}(X)$$

Which give exactly the usual unbiassed estimators. We can apply the same technique to estimate the moments of $\log(\Lambda)$

2 Moments of $\log(\Lambda)$

We are interested in estimating the variance of $\log(\lambda)$. We therefore perform a change of variable $\lambda = e^u$. This gives us

$$\int_{-\infty}^{\infty} e^{-e^u} \frac{e^{un}}{n!} \pi(e^u) e^u \, du = p_n$$

Now we have

$$\begin{aligned} e^{-e^u} e^{nu} &= \sum_{k=0}^{\infty} (-1)^k \frac{e^{(n+k)u}}{n!} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{n!} \sum_{l=0}^{\infty} \frac{((n+k)u)^l}{k!} \\ &= \sum_{l=0}^{\infty} \frac{u^l}{l!} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)^l}{k!} \end{aligned}$$

Now we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)^l}{k!} &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{m=0}^{l} \binom{l}{m} (n+k-1)^m \\ &= \sum_{m=0}^{l} \binom{l}{m} \sum_{k=0}^{\infty} \frac{(-1)^k (n+k-1)^m}{n!} \end{aligned}$$

If we let $S_{n,l} = \sum_{k=0}^{\infty} \frac{(-1)^k (n+k)^l}{k!}$, then we have shown

$$S_{n,l} = \sum_{m=0}^{l} \binom{l}{m} S_{n-1,m}$$

It is also easy to compute $S_{n,0} = e^{-1}$. We simplify things by letting $T_{n,k} = eS_{n,k}$. Finally, we will compute

$$
\begin{aligned}
T_{0,k} &= e\sum_{l=0}^{\infty}(-1)^k\frac{l^k}{l!} \\
&= e\sum_{l=0}^{\infty}(-1)^l\frac{l^k}{l!} \\
&= e\sum_{l=0}^{\infty}(-1)^l\frac{l^{k-1}}{(l-1)!} \\
&= e\sum_{l=0}^{\infty}\frac{(-1)^l}{(l-1)!}\sum_{m=0}^{k-1}\binom{k-1}{m}(l-1)^m \\
&= -e\sum_{m=0}^{k-1}\binom{k-1}{m}\sum_{l=1}^{\infty}\frac{(-1)^{l-1}(l-1)^m}{(l-1)!} \\
&= -e\sum_{m=0}^{k-1}\binom{k-1}{m}S_{0,m} \\
&= -\sum_{m=0}^{k-1}\binom{k-1}{m}T_{0,m}
\end{aligned}
$$

If we further let $R_{n,l} = \frac{T_{n,l}}{l!}$, then this recurrence becomes

$$
R_{n,l} = \sum_{m=0}^{l}\frac{R_{n-1,m}}{(l-m)!}
$$

We can then compute the following table of values of $R_{n,k}$

		k					
		0	1	2	3	4	5
n	0	1	-1	0	$\frac{1}{6}$	$\frac{1}{24}$	$-\frac{1}{60}$
	1	1	0	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{40}$
	2	1	1	0	$-\frac{1}{2}$	$-\frac{7}{24}$	0
	3	1	2	$\frac{3}{2}$	$\frac{1}{6}$	$-\frac{7}{12}$	$-\frac{59}{120}$
	4	1	3	4	$\frac{17}{6}$	$\frac{17}{24}$	$-\frac{13}{20}$
	5	1	4	$\frac{15}{2}$	$\frac{17}{2}$	$\frac{73}{12}$	$\frac{91}{40}$

We can then compute the following table of values of $T_{n,k}$

	$k=0$	1	2	3	4	5
$n=0$	1	-1	0	1	1	-2
1	1	0	-1	-1	2	9
2	1	1	0	-3	-7	0
3	1	2	3	1	-14	-59
4	1	3	8	17	17	-78
5	1	4	15	51	146	273

We can now compute the LU factorisation:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 2 & 0 & 0 & 0 & \cdots \\ 1 & 3 & 6 & 6 & 0 & 0 & \cdots \\ 1 & 4 & 12 & 24 & 24 & 0 & \cdots \\ 1 & 5 & 20 & 60 & 120 & 120 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 & 1 & 1 & -2 & \cdots \\ 0 & 1 & -1 & -2 & 1 & 11 & \cdots \\ 0 & 0 & 1 & 0 & -5 & -10 & \cdots \\ 0 & 0 & 0 & 1 & 2 & -5 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The lower triangular part of this is easily seen to have entries

$$L_{ij} = \frac{i!}{(i-j)!}$$

and inverse

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & 0 & 0 & \cdots \\ \frac{1}{2} & -1 & 1 & 0 & 0 & 0 & \cdots \\ -\frac{1}{6} & \frac{1}{2} & -1 & 1 & 0 & 0 & \cdots \\ \frac{1}{24} & -\frac{1}{6} & \frac{1}{2} & -1 & 1 & 0 & \cdots \\ -\frac{1}{120} & \frac{1}{24} & -\frac{1}{6} & \frac{1}{2} & -1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This gives us

$$q_n = \sum_{k=0}^{n} (-1)^{n-k} \frac{k!}{(n-k)!} p_k$$

Since for any empirical dataset, we must have $p_k = 0$ for all sufficiently large k, this means that

$$q_n = O\left(\frac{1}{(n-m)!}\right)$$

for some fixed m. In particular, q_n converges to zero very quickly, so we can find a solution to $UDx = q$ by solving $x = D^{-1}U^{-1}q$. We compute

$$U^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 2 & 0 & 9 & \cdots \\ 0 & 0 & 1 & 0 & 5 & -15 & \cdots \\ 0 & 0 & 0 & 1 & -2 & 15 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and find that $(U^{-1})_{ij}$ is given by the recurrence

$$(U^{-1})_{ij} = (U^{-1})_{i,j-1} - \sum_{k=i-1}^{j-1} (-1)^{k-i} \binom{k}{i-1} (U^{-1})_{k,j-1}$$

We can now calculate $T^{-1} = U^{-1}L^{-1}$. which gives

$$\begin{aligned} (T^{-1})_{ij} &= \sum_{k=i}^{\infty} (U^{-1})_{ik}(L^{-1})_{kj} \\ &= \sum_{k=j}^{\infty} (U^{-1})_{ik} \frac{(-1)^{k-j}}{j!(k-j)!} \end{aligned}$$

If we let $V_{ij} = \frac{1}{j!}(U^{-1})_{ij}$, then we have
We factorise

$$U^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 0 & 0 & 1 & 3 & 6 & 10 & \cdots \\ 0 & 0 & 0 & 1 & 4 & 10 & \cdots \\ 0 & 0 & 0 & 0 & 1 & 5 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & -2 & 9 & -36 & \cdots \\ 0 & 1 & -1 & 5 & -20 & 100 & \cdots \\ 0 & 0 & 1 & -3 & 17 & -100 & \cdots \\ 0 & 0 & 0 & 1 & -6 & 15 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -10 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

In particular, we define $q = L^{-1}p$, and define $\mu = U^{-1}q$. We also find that $(U^{-1})_{ij} = (-1)^{i+j}V_{ij}$, where $V_{i,j}$ is given by the recurrence:

$$V_{i,j} = V_{i-1,j-1} - V_{i,j-1} + \sum_{k=i}^{j-1} V_{k,j-1} \binom{k+1}{i}$$

Finally, set

$$Q = \begin{pmatrix} 1 & 0 & 1 & -2 & 9 & -36 & \cdots \\ 0 & 1 & -1 & 5 & -20 & 100 & \cdots \\ 0 & 0 & 1 & -3 & 17 & -100 & \cdots \\ 0 & 0 & 0 & 1 & -6 & 15 & \cdots \\ 0 & 0 & 0 & 0 & 1 & -10 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and let $r = Qq$. we are interested in estimating the underlying variance of U, which is given by $\mu_2 - {\mu_1}^2$. We have $\mu_2 = \sum_i \frac{i(i-1)}{2} r_i$ and $\mu_1 = \sum_{i,j} iQ_{ij}q_j$. We therefore get

$$\mu_2 - {\mu_1}^2 = \sum_i \frac{i(i-1)}{2} r_i - \sum_{i,j} ijr_ir_j$$

We can compute T^{-1} as follows:

		k 0	1	2	3	4	5
n	0	1	0	0	0	0	0
	1	1	−1	0	0	0	0
	2	−1	0	1	0	0	0
	3	$\frac{1}{2}$	−1	−2	−4	0	0
	4	$\frac{4}{31}$	0	3	12	31	0
	5	1	−1	−4	−22	−98	−306

We now have

$$e^{nu - e^u} = \sum_{k=0}^{\infty} \frac{S_{n,k}}{k!} u^k$$

Now we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{-e^u} e^{u(n+1)}}{n!} \pi(e^u) e^u \, du &= p_n \\ \int_{-\infty}^{\infty} \frac{1}{n!} \sum_{k=0}^{\infty} \frac{S_{n,k}}{k!} u^k \pi(e^u) e^u \, du &= p_n \\ \sum_{k=0}^{\infty} \frac{S_{n,k}}{n!k!} \int_{-\infty}^{\infty} u^k \pi(e^u) e^u \, du &= p_n \\ \sum_{k=0}^{\infty} \frac{S_{n,k}}{k!} \mathbb{E}\left(U^k\right) &= n! p_n \end{aligned}$$

In infinite dimensional matrix terms, we are trying to solve $R\mu = q$.

We can now seek to invert this system of equations to calculate the moments of U.

We can attempt to directly relate these to the falling moments of X. We have that the kth falling moment of X is

$$k! \sum_{n=0}^{\infty} \binom{n}{k} p_n$$

If we fix l, we see that our recurrence relation for T gives a polynomial form in n. We can prove this by recurrence. Suppose that $T_{n,l} = \sum_{m=0}^{l} C_{l,m} n^m$, then we have

$$\begin{aligned}
T_{n,l} &= \sum_{m=0}^{l} \binom{l}{m} T_{n-1,m} \\
&= \sum_{m=0}^{l} \binom{l}{m} \sum_{k=0}^{m} C_{m,k} (n-1)^k \\
&= \sum_{k=0}^{l} \left(\sum_{m=k}^{l} \binom{l}{m} C_{m,k} \right) (n-1)^k \\
\sum_{k=0}^{l} C_{l,k} n^k &= \sum_{k=0}^{l} \left(\sum_{m=k}^{l} \binom{l}{m} C_{m,k} \right) \left(\sum_{i=0}^{k} \binom{k}{i} n^i \right) \\
&= \sum_{i=0}^{l} \left(\sum_{k=i}^{l} \sum_{m=k}^{l} (-1)^{k-i} \binom{l}{m} C_{m,k} \binom{k}{i} \right) n^i
\end{aligned}$$

Which gives us the equations

$$\begin{aligned}
C_{l,i} &= \sum_{k=i}^{l} (-1)^{k-i} \binom{k}{i} \sum_{m=k}^{l} \binom{l}{m} C_{m,k} \\
&= \sum_{m=i}^{l} \sum_{k=i}^{m} (-1)^{k-i} \binom{k}{i} \binom{l}{m} C_{m,k}
\end{aligned}$$

We can check that this is satisfied by $C_{l,i} = K_{l-i}\binom{l}{i}$. Now since we also have that $C_{l,0} = T_{l,0}$, which we know are the sequence of complementary Bell numbers, this gives us the triangle C. Thinking again in terms of matrices, if we let $P_{n,m} = n^m$, then we have $T_{n,l} = \sum_m C_{l,m} P_{n,m} = (PC^T)_{n,l}$, so in matrix form we have $T = PC^T$, and thus $T^{-1} = (C^T)^{-1} P^{-1}$. Since C is triangular, we can compute its inverse directly by Gaussian elimination. For the Vandermonde matrix P,