MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 2
Due in: Monday 25th September, 11:30 AM

On this sheet, all sequences are sequences of real numbers. Please hand in solutions to questions 1-3. Question 4 is for interest only – feel free to collaborate on it or ask me about it.

Compulsory questions

1 Which of these series converge? In each case, determine whether the convergence is absolute. Justify your answers.

(a) $\sum_{n=0}^{\infty}(-1)^n \frac{n^2+3}{n^3-7n+4}$

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\arctan n}$

(c) $\sum_{n=1}^{\infty}(-1)^n \left(1-cos\left(\frac{1}{n}\right)\right)$ [Hint: you'll need to use a polynomial approximation to $\cos\theta$. You may use $|\sin\theta| \leqslant |\theta|$ to prove your approximation.]

(d) $\sum_{n=0}^{\infty}(-1)^n \frac{2+(-1)^n}{n}$

(e) $\sum_{n=1}^{\infty} a_n$ where $a_n = \begin{cases} \frac{2}{m} & \text{if } n = m^2 \\ \frac{-1}{n} & \text{if } n \text{ is not a perfect square} \end{cases}$

2 We showed (Theorem 6.18) that if $\sum_{n=0}^{\infty} a_n$ converges conditionally then $\sum_{n=0}^{\infty} a_n^+$ and $\sum_{n=0}^{\infty} a_n^-$ both diverge. Show that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} a_n^-$ both diverge, but the sequence $a_n \to 0$ as $n \to \infty$ then there is a (conditionally) convergent rearrangement of $\sum_{n=0}^{\infty} a_n$.

3 (a) Suppose $\sum_{n=0}^{\infty} a_n$ converges to x, and furthermore, suppose that the partial sums $S_k = \sum_{n=0}^{k} a_n$ are such that $\sum_{n=0}^{\infty}\left|x - S_k\right|$ converges. Prove that $\sum_{n=0}^{\infty} a_n$ converges absolutely. [Hint: use the triangle inequality ($|a + b| \leqslant |a| + |b|$) and the comparison test.]

(b) If $\sum_{n=0}^{\infty}\left|x - S_k\right|$ diverges, (Where, as in (a), $\sum_{n=0}^{\infty} a_n = x$ and $S_k = \sum_{n=0}^{k} a_n$) must the convergence of $\sum_{n=0}^{\infty} a_n$ be conditional? Give a proof or a counterexample.

Optional questions

4 (a) If $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty}(a_n)^3$ converges, must $\sum_{n=0}^{\infty}(a_n)^5$ converge?

(b) If $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty}(a_n)^5$ converges, must $\sum_{n=0}^{\infty}(a_n)^3$ converge?