MATH 3090, Advanced Calculus I
Fall 2006
Toby Kenney
Homework Sheet 3
Due in: Monday 2nd October, 11:30 AM

Please hand in solutions to questions 1-3. Question 4 is for interest only – feel free to collaborate on it or ask me about it.

Compulsory questions

1 Define the sequence a_n recursively by $a_0 = 1$, and $a_n = \sum_{i=1}^{n} \frac{2a_{n-i}}{(i+2)!}$ for $n \geqslant 1$. Given that $\sum_{n=0}^{\infty} a_n$ converges, show that $\sum_{n=0}^{\infty} a_n = \frac{1}{2(3-e)}$. [Hint: Take the Cauchy product with the series $\sum_{n=0}^{\infty} \frac{1}{(n+2)!}$. Now use the relation $a_n = \sum_{i=1}^{n} \frac{2a_{n-i}}{(i+2)!}$ to simplify. The result should look similar to $\sum_{n=0}^{\infty} a_n$, and enable you to calculate it.]

(You might like to try proving that $\sum_{n=0}^{\infty} a_n$ converges. To do this, compare it to $C\alpha^{-n}$, where α satisfies $\frac{e^{\alpha}-1-\alpha}{\alpha^2} = 1$ – assume the terms a_i for $i < n$ are all $< C\alpha^{-i}$ then use the recursive definition of a_n.)

2 For each of the following functions, calculate the pointwise limit, f, if it exists, and determine whether the convergence is uniform. If no domain is specified, the f_n are functions on the whole of $\mathbb{R}$.

(a) $f_n(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - nx & \text{if } 0 \leqslant x \leqslant \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \end{cases}$

(b) $f_n(x) = x^n e^{-nx^2}$

(c) $f_n(x) = \sin\left(\frac{x}{n}\right)$

(d) $f_n(x) = \sin(nx)$

(e) $f_n(x) = x^n$ for x in the interval $(0, 1)$ (endpoints not included).

(f) $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{p}{n^q} \text{for integers } p \text{ and } q \\ 0 & \text{otherwise} \end{cases}$

3 Let f_n be a sequence of continuous functions converging uniformly to f (which is therefore continous). Suppose that $x_n \to x$ is a sequence of real numbers. Show that $f_n(x_n) \to f(x)$. (You may assume that if f is continuous and $a_n \to a$, then $f(a_n) \to f(a)$.) [Hint: for $\epsilon > 0$, first choose N so that for $n > N$, $|f(x_n) - f(x)| < \frac{\epsilon}{2}$, then choose $M > N$ so that $|f_M - f| < \frac{\epsilon}{2}$. Do not choose M before N – it won't work!]

Optional questions

4 (a) Is there an equivalent result to the Bolzano-Weierstrass theorem for functions – i.e. given a sequence of functions $f_n : \mathbb{R} \to [0, 1]$, must it have a uniformly convergent subsequence?

(b) Prove that if (f_n) is a sequence of functions satisfying

$$(\forall \epsilon > 0)(\exists N)(\forall n, m \geqslant N)(\forall x)(|f_n(x) - f_m(x)| < \epsilon)$$

then (f_n) converges uniformly to some limit f.