MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 4
Due in: Wednesday 11th October, 11:30 AM

Please hand in solutions to questions 1-3. Question 4 is for interest only – feel free to collaborate on it or ask me about it.

Compulsory questions

1 Which of the following series of functions converge uniformly on the interval (0,1)? If they do not converge uniformly, is the limit continuous?

(a) $\sum_{n=0}^{\infty} \frac{x^n}{x+n}$ [You may assume that $\left(1-\frac{1}{N}\right)^N \geqslant \frac{1}{12}$ for $N \geqslant 2$.]

(b) $\sum_{n=1}^{\infty} \frac{1}{n^2+x}$

(c) $\sum_{n=1}^{\infty} \frac{\cos(n(x+1))}{n}$ [Hint: multiply by $2\sin\left(\frac{x+1}{2}\right)$. Recall that $2\sin\alpha\cos\beta = \sin(\beta+\alpha) - \sin(\beta-\alpha)$. There should then be cancellation between consecutive terms of the resulting series.]

2 (a) Suppose (f_n) is a sequence of continuously differentiable functions on an interval $[a,b]$, converging pointwise to f. Suppose the derivatives f_n' converge uniformly to g on $[a,b]$. (In Theorem 7.12 we showed that g is the derivative of f.) Prove that $f_n \to f$ uniformly on $[a,b]$. (You may assume that $\left|\int_x^y f(t)dt\right| \leqslant \int_x^y |f(t)|dt$.)

(b) What if instead of the finite interval $[a,b]$, the sequence f_n converges pointwise to f on the interval $[a,\infty)$, and $f_n' \to g$ uniformly on $[a,\infty)$?

3 Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence?

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n^3+2n+3}$

(b) $\sum_{n=0}^{\infty} \frac{x^{\left(n^2\right)}}{n!}$

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n(n+3)}$

Optional questions

4 If the coefficients a_n are all required to be integers, what are the possible values for the radius of convergence of $\sum_{n=0}^{\infty} a_n$?