MATH 3090, Advanced Calculus I

Fall 2006

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Homework Sheet 5

Due in: Monday 16th October, 11:30 AM

Please hand in solutions to questions 1-4. Question 5 is for interest only – feel free to collaborate on it or ask me about it.

Compulsory questions

1 (a) Find the radius of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$.

(b) Evaluate $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^{n+1}}$ on the interval $(-R, R)$, where R is the radius of convergence [Hint: it's a geometric series]. On what interval is the function you get infinitely differentiable?

2 Let

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Differentiate $f(x)$. Show that $\frac{f(x)}{x^n} \to 0$ as $x \to 0$, for any n. Can f be expressed as a power series about 0?

3 Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, and suppose $x_0 \in (-R, R)$. Show that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a Taylor series expansion about x_0 with radius of convergence at least $R - |x_0|$.

[Hint: Calculate the coefficients as power series in x_0 by differentiating the series repeatedly. Now observe that $\sum_{n=0}^{\infty} |a_n| \left(\sum_{m=0}^{n} \begin{pmatrix} n \\ m \end{pmatrix} |x_0|^{n-m} |x - x_0|^m \right)$ converges when $|x - x_0| < R - |x_0|$. Therefore, we can rearrange the terms to get that $\sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} |a_n| \begin{pmatrix} n \\ m \end{pmatrix} |x_0|^{n-m} |x - x_0|^m \right)$ converges. Compare this to the Taylor series we got by differentiating at x_0.]

4 Find power series about 0 for the following integrals:

(a) $\int_{t=0}^{x} \cos(t^3) dt$

(b) $\int_{t=0}^{x} \frac{e^t - 1}{t} dt$

Optional questions

5 Show that if $\sum_{n=0}^{\infty} a_n x^n$ has positive radius of convergence, then for any $\epsilon > 0$, we can find $\delta > 0$ such that $\sum_{n=1}^{\infty} a_n x^n < \epsilon$ for all $x < \delta$. Deduce that we can find $\delta > 0$ such that $\sum_{n=2}^{\infty} a_n x^n < \epsilon x$ [Hint: consider the

series $\sum_{n=1}^{\infty} a_{n+1}x^n$]. Deduce that if f has a Taylor series about every $x \in [a,b]$, and $f'(x) < C$ for all $x \in [a,b]$, then $|f(b) - f(a)| \leqslant C(b-a)$. [Hint: show it is less than $(C+\epsilon)(b-a)$ for any $\epsilon > 0$.]