MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 6
Due in: Monday 6th November, 11:30 AM

Please hand in solutions to questions 1-4. Question 5 is for interest only – feel free to collaborate on it or ask me about it.

Compulsory questions

1 Show that $\cos n\theta = \sum_{j=0}^{\frac{n}{2}}(-1)^j \begin{pmatrix} n \\ 2j \end{pmatrix} \cos^{n-2j}\theta \sin^{2j}\theta$ and that $\sin n\theta = \sum_{j=0}^{\frac{n}{2}}(-1)^j \begin{pmatrix} n \\ 2j+1 \end{pmatrix} \cos^{n-2j-1}\theta \sin^{2j+1}\theta$. [Hint: use $e^{i\theta} = \cos\theta + i\sin\theta$ and the binomial formula.]

2 Which non-zero complex numbers z have the property that $z + \frac{1}{z}$ is real?

3 Evaluate the following improper integrals

(a) $\int_0^\infty \int_t^\infty e^{-x^2} dxdt$

(b) $\int_0^\infty \frac{1-\cos t}{t^2} dt$ [Hint: you can calculate $\int_0^\infty \frac{\sin xt}{t} dt$ by the change of variable $u = xt$. Now integrate with respect to x.]

4 Do the following series converge? Justify your answers.

(a) $\sum_{n=1}^\infty \frac{1\times 5\times 9\times\cdots\times(4n+1)}{3\times 7\times 11\times\cdots\times(4n+3)}$

(b) $\sum_{n=1}^\infty \frac{(2n)!^4}{(4n)!(n!)^4}$

Optional questions

5 Define $I_\alpha(f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} f(t)dt$. $I_\alpha(f)$ is called the αth-order fractional integral of f.

(a) Show that the derivative of $I_{\alpha+1}(f)$ is $I_\alpha(f)$ for $\alpha > 0$, and that the derivative of $I_1(f)$ is f.

(b) Show that $I_\alpha\left(I_\beta(f)\right) = I_{\alpha+\beta}(f)$ for any $\alpha, \beta > 0$.