MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 8
Due in: Monday 20th November, 11:30 AM

Please hand in solutions to questions 1-3.

Compulsory questions

1 Recall that the Fourier series for $f(x) = x$ when $-\pi \leqslant x < \pi$, and f 2π-periodic is $2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$. By integrating this 4 times, find the Fourier series for $g(x) = \frac{x^5}{120} - \frac{\pi^2 x^3}{36} + \frac{7\pi^4 x}{360}$. [Remember to add the constant terms.]

2 Find the Fourier sine and cosine series for the following functions on the interval $[0, \pi]$.

(a) $f(x) = e^x$ [Hint: $\cos nx = \frac{e^{inx}+e^{-inx}}{2}$, $\sin nx = \frac{e^{inx}-e^{-inx}}{2i}$.]

(b) $f(x) = \sin\left(x + \frac{\pi}{3}\right)$.

3 Define f by $f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}$. (This converges for all x by Dirichlet's test – see Corollary 6.27.)

(a) Show that the series converges uniformly on the intervals (δ, π) and $(-\pi, -\delta)$. [You may assume that the convergence in Dirichlet's test (Theorem 6.25) is uniform provided that there some bound C on the $|B_n|$ which works for all x in the interval.]

This means that f is a continuous function everywhere except perhaps at integer multiples of π. By subtracting off a multiple of the square wave ($h(x) = \begin{cases} -1 & \text{if } (2n-1)\pi < x \leqslant 2n\pi \\ 1 & \text{if } 2n\pi < x \leqslant (2n+1)\pi \end{cases}$) and a multiple of the sawtooth wave ($s(x) = x$ for $-\pi < x \leqslant \pi$, and 2π-periodic) from f, we get a function g that is continuous at all x.

(b) Show that g is not piecewise continuously differentiable. [Hint: if it were piecewise continuously differentiable, what would the Fourier coefficients have to be? (Recall that the square wave has Fourier Coefficients $b_{2n+1} = \frac{4}{(2n+1)\pi}$ and all other coefficients 0, while the sawtooth wave has Fourier coefficients $b_n = \frac{(-1)^{n+1}}{n}$.) Use Bessel's inequality to show that these cannot be the Fourier coefficients of a piecewise continuous function.]