MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 9
Due in: Monday 27th November, 11:30 AM

Please hand in solutions to questions 1-3. Where appropriate, you may quote the solutions to the heat equation and the wave equation given in lectures.

Compulsory questions

1 An rod of length π and 0 thickness is heated to a uniform 100°C at time 0. The ends of the rod are then immersed in ice, to fix their temperature at 0°C.

(a) When the temperature in the middle of the rod ($x = \frac{\pi}{2}$) reaches $\frac{400}{\pi} \times \left(\frac{1}{2} - \frac{1}{3\times 2^9}\right)^\circ$C, show that the temperature at the point one third of the way along the rod ($x = \frac{\pi}{3}$) is at most $\frac{100\sqrt{3}}{\pi}^\circ$C. [Hint: Both the temperature in the middle of the rod, and the temperature one third of the way along the rod can be expressed as alternating series (you have to combine some terms) with terms decreasing in modulus. Recall that if a series is alternating and its terms have decreasing modulus, then the partial sums give alternately lower and upper bounds for the whole sum. You should be able to show that $e^{-kt} \leqslant \frac{1}{2}$.]

(b) Show that once the $e^{-4kt} \leqslant \frac{1}{2}$, the temperature never gets below 0°C anywhere on the rod. (This is true for all positive time, but it is easier to show if we make the assumption that $e^{-4kt} \leqslant \frac{1}{2}$.) [Hint: show that by this time, the first term of the Fourier series solution is larger in modulus than the sum of the moduli of all the others. (The moduli of the other terms are bounded by a geometric series.) This term is positive for all $x \neq 0, \pi$, so the temperature must be positive for all $x \neq 0, \pi$. You may assume that $\left|\frac{\sin nx}{\sin x}\right| \leqslant n$ for all integers n.]

2 A guitar string of length π is plucked by pulling the part of the string with x coordinate a a distance 1 upwards. The equation for the string is therefore

$$f(x) = \begin{cases} \frac{x}{a} & \text{if } 0 \leqslant x \leqslant a \\ \frac{\pi - x}{\pi - a} & \text{if } a \leqslant x \leqslant \pi \end{cases}$$

where $f(x)$ is the vertical displacement of the string at point x. It is released from rest in this position at time 0 (so $u(x, 0) = f(x)$, $u_t(x, 0) = 0$).

(a) Assuming that the Fourier series solution (8.37) does indeed model the future behaviour of the string, calculate the Fourier coefficients b_n of $\cos nt \sin nx$ as functions of a.

(b) Show that $a = \frac{\pi}{2}$ gives a local maximum of the coefficient b_1. We will see next week that (assuming that $\frac{\pi}{2}$ is the global maximum) this means that the middle of the string is the best place to pluck it if we want to get as close to a pure note as possible.

3 Suppose we modify the wave equation to account for the fact that the string is not perfectly elastic. We use the equation:

$$u_{tt} = c^2 u_{xx} - 2\delta u_t$$

where δ is a small positive constant. Assume that $\delta < c$. Use separation of variables to find a family of solutions to this equation that satisfy the boundary conditions $u(0,t) = u(\pi,t) = 0$.