MATH 3090, Advanced Calculus I
Fall 2006
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Midterm Examination
Wednesday 25th October: 18:00—19:30
Model Answers

Answer all questions.

1 Which of the following series converge? For series which converge, is the convergence absolute? Justify your answers. (You may assume convergence of geometric series and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *for* $p > 1$*, and divergence of* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *for* $p \leqslant 1$*.)*

(a) $\sum_{n=0}^{\infty} \frac{n+4}{2^n}$

The ratio of consecutive terms in the series is $\frac{(n+5)2^{n+1}}{(n+4)2^n} = \frac{n+5}{2(n+4)} \to \frac{1}{2}$ as $n \to \infty$. Therefore, by the ratio test, the series converges absolutely.

(b) $\sum_{n=1}^{\infty} \sqrt{n+\frac{1}{n}} - \sqrt{n}$

$\left(\sqrt{n+\frac{1}{n}} - \sqrt{n}\right)\left(\sqrt{n+\frac{1}{n}} + \sqrt{n}\right) = n + \frac{1}{n} - n = \frac{1}{n}$, so

$$\left(\sqrt{n+\frac{1}{n}} - \sqrt{n}\right) = \frac{1}{n\left(\sqrt{n+\frac{1}{n}} + \sqrt{n}\right)} \leqslant \frac{1}{n^{\frac{3}{2}}}$$

and all the terms in the series are non-negative, so the series converges absolutely by comparison to $\frac{1}{n^{\frac{3}{2}}}$.

2 Show that if $f_n \to f$ *uniformly on the interval* $[a, b]$*, and all the* f_n *are continuous on* $[a, b]$*, then* f *is continuous on* $[a, b]$*.*

We need to show that given $\epsilon > 0$, and $x \in [a, b]$, there is a $\delta > 0$, such that $(\forall y \in [a, b])(|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon)$. Since the $f_n \to f$ uniformly on $[a, b]$, we can find an N such that $(\forall x \in [a, b])(|f_N(x) - f(x)| < \frac{\epsilon}{3})$ (indeed we can find N such that this holds for every $n \geqslant N$). Now, f_N is continuous, so we can choose a $\delta > 0$, such that

$$(\forall y \in [a, b])(|y - x| < \delta \Rightarrow |f_N(y) - f_N(x)| < \frac{\epsilon}{3})$$

Now if $|x - y| < \delta$ then

$$|f(x) - f(y)| = |f(x) - f_N(x) + f_N(x) - f_N(y) + f_N(y) - f(y)| \leqslant$$
$$|f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

which is what we needed for f to be continuous.

3 *Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence? Justify your answers.*

(a) $\sum_{n=0}^{\infty} \frac{x^n}{2^{n^2}}$

The ratio of consecutive terms is $\frac{x^{n+1}2^{(n+1)^2}}{x^n 2^{n^2}} = \frac{x}{2^{2n+1}}$, but $\frac{x}{2^{2n+1}} \to 0$ as $n \to \infty$, for any x, so the radius of convergence is infinite.

(b) $\sum_{n=0}^{\infty} \frac{x^{3n+1}}{8^n(n+3)}$

The ratio of consecutive terms is $\frac{x^{3n+4}8^n(n+3)}{x^{3n+1}8^{n+1}(n+4)} = \frac{x^3(n+3)}{8(n+4)} \to \frac{x^3}{8}$ as $n \to \infty$, and $\left|\frac{x^3}{8}\right| < 1$ whenever $|x| < 2$, and $\left|\frac{x^3}{8}\right| > 1$ whenever $|x| > 2$. Therefore, the radius of convergence is 2. When $x = 2$, the series diverges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n}$. When $x = -2$, it converges by the alternating series test.

4 *What is a Cauchy sequence? Prove that Cauchy sequences of real numbers are convergent. (You may assume the Bolzano-Weierstrass theorem.)*

A Cauchy sequence is a sequence a_n that satisfies $(\forall \epsilon > 0)(\exists N)(\forall m, n \geqslant N)(|a_n - a_m| < \epsilon)$.

Theorem 1. *Cauchy sequences of real numbers converge.*

Proof. Let a_n be a Cauchy sequence. For $\epsilon = 1$, we can pick an N so that for $m, n \geqslant N$ we have $|a_m - a_n| < 1$. In particular, we have that $a_N - 1 < a_n < a_N + 1$ for all $n > N$. Therefore, a Cauchy sequence is bounded. Thus, we can apply the Bolzano-Weierstrass theorem to deduce that a_n has a convergent subsequence $a_{n_i} \to a$. Now given $\epsilon > 0$, we can choose I so that $(\forall i \geqslant I)(|a_{n_i} - a| < \frac{\epsilon}{2})$. Also, we can choose N so that $(\forall m, n \geqslant N)(|a_n - a_m| < \frac{\epsilon}{2})$. Therefore, if $M = \max(N, n_I)$, then for $n \geqslant M$ and $n_i \geqslant M$, $|a_n - a| \leqslant |a_n - a_{n_i}| + |a_{n_i} - a| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$, so $a_n \to a$. □

5 *Which of the following series of functions of x converge uniformly on the interval (0,1)? Justify your answers.*

(a) $\sum_{n=0}^{\infty} f_n(x)$ *where* $f_n(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$

Observe that the pointwise limit of this sum is just

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \\ 0 & \text{otherwise} \end{cases}$$

Now the partial sum up to N is zero for all $x < \frac{1}{N}$, so in particular $f\left(\frac{1}{N+1}\right) - \sum_{n=0}^{N} f(n) = 1$, so the series does not converge uniformly.

(b) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

For all $x \in (0,1)$, $\frac{x^n}{n^2} < \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $(0,1)$ by the Weierstrass M-test with $M_n = \frac{1}{n^2}$