MATH 3090, Advanced Calculus I
Fall 2006
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Mock Midterm Examination

This is intended to be of a similar style to the midterm exam. Since it has about the same number of questions as the midterm, it was not possible to include questions covering all areas of the syllabus; this does **not** mean that there will not be questions on these areas in the midterm exam. This is to help you revise for the midterm exam, and carries no credit in itself.

Answer all questions.

1 Show that if the series $\sum_{n=0}^{\infty} a_n$ converges, where $a_n \geqslant 0$ for all n, then so does $\sum_{n=0}^{\infty} a_n^2$.

2 Which of the following series of functions converge uniformly on the interval (0,1)? Justify your answers.

(a) $\sum_{n=1}^{\infty} f_n(x)$ where $f_n(x) = \begin{cases} \frac{1}{n^2} & \text{if } x < \frac{1}{n} \\ 0 & \text{if } x \geqslant \frac{1}{n} \end{cases}$

(b) $\sum_{n=1}^{\infty} \frac{x+\frac{1}{n^2}}{(1+x)^{n^2}}$ [Hint: substitute $y = n^2 x$, and expand the denominator]

3 Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence?

(a) $\sum_{n=0}^{\infty} \frac{x^n}{3^{2n+1}}$

(b) $\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$ (You may assume that $\left(1+\frac{1}{n}\right)^n \to e$ as $n \to \infty$.)

4 State and prove the Bolzano-Weierstrass theorem.

5 Which of the following series converge? For series which converge, is the convergence absolute? Justify your answers. (You may assume convergence of geometric series and $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$, and divergence of $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p \leqslant 1$.)

(a) $\sum_{n=1}^{\infty} (-1)^n \frac{n^2+2}{3n^2+4n+5}$

(b) $\sum_{n=1}^{\infty} \frac{\frac{\pi}{2} - \arctan n}{n}$

6 Show that if $f_n \to f$ uniformly on the interval $[a,b]$, and all the f_n are continuous on $[a,b]$, then f is continuous on $[a,b]$. If the f_n are all differentiable at some $x \in [a,b]$, must f be differentiable at x? Give a proof or a counterexample.