MATH 2051, Problems in Geometry
Fall 2007
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Mock Final Examination
Time allowed: 3 hours
Calculators not permitted.

Note that diagrams are not drawn to scale. Scale drawing does **not** constitute a proof. Justify all your answers. This mock exam contains more questions than the final exam will, particularly on hyperbolic geometry, in order to give a better idea of the range of questions that might be asked.

Answer all questions

1 *(a) Let A and B be two points. Let $0 < \lambda < 1$ be a real number. Let P be a point such that $\frac{AP}{BP} = \lambda$. Let $\theta = \angle ABP$. Use the cosine rule on triangle ABP to find a quadratic equation satisfied by BP.*

We know that $AP^2 = \lambda^2 BP^2$. By the cosine rule on $\triangle ABP$, we get $\lambda^2 BP^2 = BP^2 + AB^2 - 2AB.BP\cos\theta$, or $(\lambda^2-1)BP^2 + (2AB\cos\theta)BP - AB^2 = 0$.

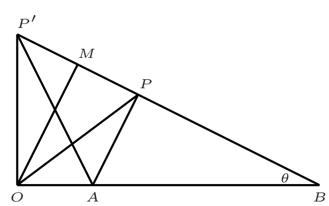
(b) Since a quadratic equation has at most two solutions, there is at most one other point P' on the line BP such that $\frac{AP'}{BP'} = \lambda$. Show that $\frac{BP+BP'}{2} = \left(\frac{AB}{1-\lambda^2}\right)\cos\theta$ and $\frac{BP-BP'}{2} = \left(\frac{AB}{1-\lambda^2}\right)\sqrt{\cos^2\theta - (1-\lambda^2)}$.

We can use the quadratic formula to get $BP = \frac{-2AB\cos\theta \pm \sqrt{4AB^2\cos^2\theta + 4AB^2(\lambda^2-1)}}{2(\lambda^2-1)}$.

We can add the two solutions to get $BP + BP' = 2\left(\frac{-AB}{\lambda^2-1}\right)\cos\theta$.

We can also take the difference between the two solutions to get $BP - BP' = 2\frac{\sqrt{4AB^2\cos^2\theta+4AB^2(\lambda^2-1)}}{2(\lambda^2-1)} = 2\left(\frac{AB}{(1-\lambda^2)}\right)\sqrt{\cos^2\theta - (1-\lambda^2)}$

(c) Let O be the point on AB extended past A, such that $OB = \frac{AB}{1-\lambda^2}$. Show that P and P' both lie on a circle centre O, radius $\frac{AB\lambda}{1-\lambda^2}$. [Hint: Let M be the midpoint of P and P'; show that OM is perpendicular to BP.]

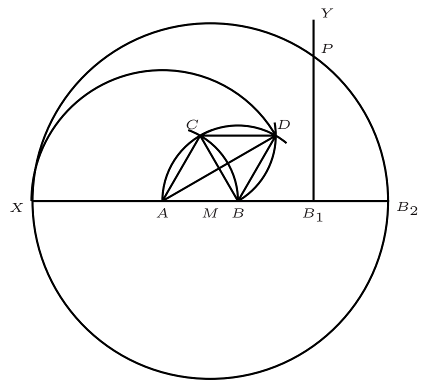


If M is the midpoint of PP', then we have that $BM = \frac{BP+BP'}{2} = \left(\frac{AB}{(1-\lambda^2)}\right)\cos\theta = OB\cos\theta$. This means that $\triangle OMB$ is right-angled at M (since if D is the foot of the perpendicular from O to BP, then $BD = OB\cos\theta = BM$). Now we can use Pythagoras' theorem to get

$OP^2 = OM^2 + MB^2 = OB^2 \sin^2\theta + \left(\frac{BP - BP'}{2}\right)^2$. Using the result in (b), this gives $OP^2 = OB^2\left(\sin^2\theta + \cos^2\theta - (1-\lambda^2)\right) = OB^2\lambda^2$. Therefore, $OP = OB\lambda = \frac{AB\lambda}{1-\lambda^2}$

2 *Given a line segment of length 1, describe how to construct a line segment of length $\sqrt{2+\sqrt{3}}$ using just a straight-edge and a pair of compasses. [You do not need to prove that your construction works.]*

We start by constructing a line segment of length $\sqrt{3}$. The easiest way to do this is just to construct an equilateral triangle ABC with base AB by letting C be a point where the circle centre A passing through B and the circle centre B passing through A intersect. Similarly, we construct the other equilateral triangle BCD with base BC. Now $AD = \sqrt{3}$.



Now we extend the line AB in both directions. Let X be the point on this line on the opposite side of A to B, such that $AX = \sqrt{3}$ (we find this point by drawing the circle centre A passing through D and seeing where it meets AB). Now we find the points B_1 and B_2 on the line AB, on the opposite side of B to A, such that $BB_1 = B_1B_2 = 1$ (by drawing the circle centre B passing through A, and the circle centre B_1 through B). Now if we draw the circles centre B_2 passing through B and centre B passing through B_2, and let Y be a point where they intersect, then we draw the line B_1Y, which is parallel to AB. Finally, we construct the midpoint of XB_2 by drawing circles centre X passing through B_2 and centre B_2 passing through X, and joining the points Z_1 and Z_2 where they intersect. Let M be the point where W_1W_2 meets AB. Now we draw the circle centre M, passing through X. Let P be the point where this circle meets B_1Y. B_1P is a line segment of length $\sqrt{2+\sqrt{3}}$.

3 *(a) Show that the hyperbolic distance from the origin to the point x, for a positive real number x is $2\tanh^{-1} x$.*

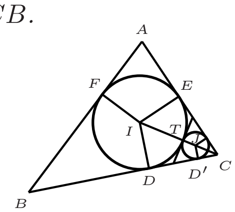
The hyperbolic distance from $(0,0)$ to $(0,x)$ is given by $\int_0^x \frac{2dy}{1-y^2}$ (since the x-coordinate is constantly 0. Make the substitution $y = \tanh z$. This gives

$\frac{dy}{dz} = \frac{1}{\cosh^2 x} = 1 - \tanh^2 x$. Therefore, the integral is $\int_0^{\tanh^{-1} x} \frac{2(1-tanh^2 z)}{1-tanh^2 z} dz = 2\tanh^{-1} x$.

(b) Deduce that the hyperbolic distance from z to w is $2\tanh^{-1}\left|\frac{z-w}{\overline{w}z-1}\right|$.

We apply the isometry $x \mapsto \frac{x-w}{\overline{w}x-1}$, which sends w to 0 and z to $\frac{z-w}{\overline{w}z-1}$. Next we apply a rotation about the origin to send $\frac{z-w}{\overline{w}z-1}$ to $\left|\frac{z-w}{\overline{w}z-1}\right|$. Since the maps we applied are isometries, the hyperbolic distance from z to w is the same as the hyperbolic distance from 0 to $\left|\frac{z-w}{\overline{w}z-1}\right|$, which is $2\tanh^{-1}\left|\frac{z-w}{\overline{w}z-1}\right|$.

4 *Let ABC be a triangle with incentre I and inradius r. Let γ be a circle inside the triangle tangent to the sides AC and BC, and externally tangent to the incircle of $\triangle ABC$ (i.e. the incircle of $\triangle ABC$ and γ meet at a point T, where they have a common tangent, and the rest of γ lies outside the incircle). Let γ have radius r' and centre J. Show that $\frac{r'}{r} = \frac{1-\sin\frac{C}{2}}{1+\sin\frac{C}{2}}$, where C is the angle $\angle ACB$.*



Let the foot of the perpendicular from I to BC be D, and the foot of the perpendicular from J to BC be D'. Triangles IDC and $JD'C$ are both right angled triangles with angle θ, so they are similar, and the ratio between them is $\frac{r}{r'}$. Also, T lies on the line through I and J, which also passes through C. We note that $\frac{CT}{CJ} = \frac{CJ+r'}{CJ} = \frac{CJ+CJ\sin\frac{C}{2}}{CJ} = 1+\sin\frac{C}{2}$. Similarly, $\frac{CT}{CI} = \frac{CI-r}{CI} = \frac{CI-CI\sin\frac{C}{2}}{CI} = 1-\sin\frac{C}{2}$. Therefore, $\frac{r'}{r} = \frac{CJ}{CI} = \frac{1-\sin\frac{C}{2}}{1+\sin\frac{C}{2}}$.

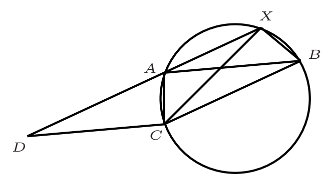
5 *Find the area of the hyperbolic triangle with vertices at 0, $\sqrt{\frac{\sqrt{3}-1}{\sqrt{3}+1}}$ and $\sqrt{\frac{\sqrt{3}-1}{\sqrt{3}+1}}i$. [Hint: $\tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$.]*

The area of an hyperbolic triangle with angles α, β and γ radians is $\pi-\alpha-\beta-\gamma$, so we just need to find the angles of this triangle. The angle at the origin is clearly $\frac{\pi}{2}$ radians. Let $x = \sqrt{\frac{\sqrt{3}-1}{\sqrt{3}+1}}$. To find the angle at x, we apply $z \mapsto \frac{z-x}{\overline{x}z-1}$, to send x to the origin. This sends 0 to x and xi to $\frac{x(i-1)}{ix^2-1} = \frac{x(i-1)(-1-ix^2)}{1+x^4} = \frac{x(1+x^2-i(1-x^2))}{1+x^4}$.

The angle between the images of the points 0 and xi from the origin is therefore $\tan^{-1}\left(\frac{1-x^2}{1+x^2}\right)$. Now $1-x^2 = \frac{(\sqrt{3}+1)-(\sqrt{3}-1)}{\sqrt{3}+1} = \frac{2}{\sqrt{3}+1}$, and

$1+x^2 = \frac{(\sqrt{3}+1)+(\sqrt{3}-1)}{\sqrt{3}+1} = \frac{2\sqrt{3}}{\sqrt{3}+1}$, so $\frac{1-x^2}{1+x^2} = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} = \tan\frac{\pi}{6}$. Therefore the angle at x in the original triangle is $\frac{\pi}{6}$. The triangle is symmetric about the line $a+ai$ (the line at 45° to the real axis), so the angle at xi is also $\frac{\pi}{6}$. Therefore, the area of the triangle is $\pi - \frac{\pi}{2} - \frac{\pi}{6} - \frac{\pi}{6} = \frac{\pi}{6}$.

6 *Let Γ be a circle. Let A and B be points on Γ. Let C be a point on Γ, and D a point outside Γ such that $ABCD$ is a parallelogram. Extend the line DA to meet Γ again at X. Show that $BX = AC$ and $AB = CX$.*



First we note that $\angle ABC = \angle BAX$ by alternate angles. However $\angle BAX = \angle BCX$ by angles in the same segment. Also, $\angle BAC = \angle BXC$ by angles in the same segment. Therefore, $\angle ACB = \angle XBC$ by angles in a triangle, so triangles ACB and XBC are similar, so since they share the side BC, they are congruent. Therefore, $AC = BX$ and $AB = CX$.

7 *There is a semiregular polyhedron with 2 square faces, one triangular face, and one pentagonal face meeting at each vertex. How many:*

(i) triangular faces

(ii) square faces

(iii) pentagonal faces

(iv) vertices

(v) edges

does it have?

Let F_3 be the number of triangular faces, F_4 the number of square faces and F_5 the number of pentagonal faces. Since there is exactly one triangular face at each vertex, and 3 vertices on each triangular face, $V = 3F_3$. Similarly, $V = \frac{4F_4}{2} = 2F_4$ and $V = 5F_5$. We therefore have $F_4 = \frac{5}{2}F_5$ and $F_3 = \frac{5}{3}F_5$. Finally, since each triangular face has 3 edges, each square face has 4 edges and each pentagonal face has 5 edges, but each edge is shared by two faces, we get $2E = 3F_3 + 4F_4 + 5F_5 = 5F_5 + 10F_5 + 5F_5 = 20F_5$. We also have $F = F_3 + F_4 + F_5 = \frac{5}{3}F_5 + \frac{5}{2}F_5 + F_5 = \frac{31}{6}F_5$ and $V = 5F_5$, so by Euler's formula, $\frac{31}{6}F_5 + 5F_5 - 10F_5 = 2$, so $(31+30-60)F_5 = 12$, i.e.

(iii) $F_5 = 12$.

This gives:

(i) $F_3 = 20$.

(ii) $F_4 = 30$.

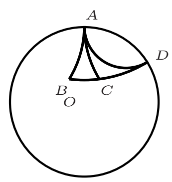
(iv) $V = 60$.

(v) $E = 120$.

8 *Show that the area of an hyperbolic triangle with angles α, β, and γ is $\pi - \alpha - \beta - \gamma$. [You may use the fact that the area of a doubly asymptotic triangle with angle θ is $\pi - \theta$.]*

We start by considering an asymptotic triangle with angles α, β and 0. We show that this triangle has area $\pi - \alpha - \beta$:

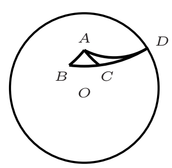
Let ABC be a singly asymptotic triangle, with A a point on the boundary of the disc. Extend BC past C to meet the boundary of the disc at D.



Now the area of the singly asymptotic triangle ABC is the area of the doubly asymptotic triangle ABD minus the area of the doubly asymptotic triangle ACD. The triangle ABD has angle α, while the triangle ACD has angle $\pi - \beta$. Therefore the area of triangle ABD is $\pi - \alpha$, and the area of triangle ACD is $\pi - (\pi - \beta) = \beta$. The area of triangle ABC is therefore $\pi - \alpha - \beta$.

Now we can show that a triangle ABC with angles α, β and γ has area $\pi - \alpha - \beta - \gamma$:

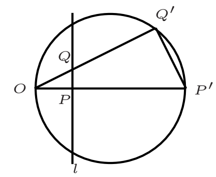
Extend BC past C to meet the boundary of the disc at D.



The area of ABC is the area of ABD minus the area of ACD. Let $\angle CAD = \delta$. Then the area of $\triangle ABD$ is $\pi - \alpha - \beta - \delta$, and the area of $\triangle ACD$ is $\pi - (\pi - \gamma) - \delta = \gamma - \delta$, so the area of $\triangle ABC$ is $\pi - \alpha - \beta - \gamma$.

9 *(a) Show that inversion in a circle sends lines not passing through the centre of the circle to circles passing through the centre of the circle.*

Let O be the centre of the circle, and let l be a straight line not through O. Drop the perpendicular from O to l, and let the foot of this perpendicular be P. Let its image under the inversion be P'. Let Q be another point on l, and let Q' be its image.



We know that $OP.OP' = OQ.OQ' = r^2$, so P, P', Q, and Q' are concyclic. Since $\angle OPQ = 90°$, we get that $\angle OQ'P = 90°$. Therefore, Q' lies on the circle with diameter OP'. This circle is therefore the image of l.

(b) What are hyperbolic straight lines in the disc model? Prove your answer. [You may use the hyperbolic isometries taught in class without proof. You may also assume that the real axis is an hyperbolic straight line.]

We can find all hyperbolic straight lines by applying isometries to the real axis. By applying rotations about the origin, we get that hyperbolic straight lines through the origin are Euclidean straight lines.

The other hyperbolic isometries that we apply are the isometries of the form $z \mapsto \frac{z-a}{\overline{a}z-1}$. These can be expressed as the composite of the transformations:

$$\begin{aligned} z &\mapsto z - \frac{1}{\overline{a}} \\ z &\mapsto \frac{1-a\overline{a}}{a\overline{z}} \\ z &\mapsto \overline{z} \\ z &\mapsto z + \frac{1}{\overline{a}} \end{aligned}$$

The first of these is a translation, so it sends a line through the origin to another line. The second is an inversion in a circle centred at the origin, so it sends this line to a circle (passing through the origin). The third is a reflection in the real axis, so it sends this circle to another circle. The fourth is a translation, so it sends the circle to another circle.

The image of a line through the origin under $z \mapsto \frac{z-a}{\overline{a}z-1}$ is a circle, and it contains the images of 0 and ∞, which are the points a and $\frac{1}{\overline{a}}$. These two points are on a line through the origin, since $\frac{a}{\frac{1}{\overline{a}}}$ is real. The power of the origin with respect to this circle is therefore $|a|\left|\frac{1}{\overline{a}}\right| = 1$. This means that the tangents from the origin to this circle have length 1, and therefore, that this circle meets the unit circle perpendicularly.

Therefore, hyperbolic lines are straight lines through the origin and circles perpendicular to the unit circle.

10 *How many hyperfaces, faces, edges and vertices does a 4-dimensional hypercube have? Justify your answer.*

A 4-dimensional hypercube has a cube as its base, then a cube at the top, and between part of the base and the corresponding part of the top, the figure formed is a figure one dimension higher, so for example, between a square face on the base and a square face on the top, there is a cubical hyperface.

There are therefore 8 cubical hyperfaces in total – one on the base and one on the top, and one for each of the 6 faces of the bottom and the top.

Similarly, there are 24 square faces – 6 on the base, 6 on the top, and one for each edge of the base.

There are 32 edges – 12 on the base, 12 on the top, and one for each of the 8 vertices on the base.

Finally, there are 16 vertices – 8 on the base and 8 on the top.

11 *Let ABC be the triply asymptotic hyperbolic triangle with vertices at 1, $0.6+0.8i$ and w where w is the point on the boundary of the unit disc such that $0.5i$ lies on the hyperbolic line between 1 and w . Find an hyperbolic isometry sending ABC to the hyperbolic triangle with vertices at the 1, -1 and i.*

First we find an isometry sending the line from 1 to w to the real axis. Since the point $0.5i$ lies on this line, we start by applying the isometry $z \mapsto \frac{z-0.5i}{-0.5iz-1}$ to send this point to the origin. This isometry sends 1 to $\frac{1-0.5i}{-1-0.5i} = \frac{(1-0.5i)(-1+0.5i)}{1.25} = \frac{3+4i}{5}$ and $0.6+0.8i$ to $\frac{6+8i-5i}{4-3i-10} = \frac{6+3i}{-6-3i} = -1$. We need to rotate about the origin to send 1 to itself. This is achieved by multiplying by $\frac{3-4i}{5}$, so that the isometry is $z \mapsto \frac{3+4i}{5}\frac{z-0.5i}{-0.5iz-1}$. This sends $\frac{3+4i}{5}$ to $\frac{-3+4i}{5}$. We now need to find an isometry that fixes the real axis and sends $\frac{-3+4i}{5}$ to i. We note that the hyperbolic line from $\frac{-3+4i}{5}$ perpendicular to the real axis is the Euclidean circle with centre $-\frac{5}{3}$ and radius $\frac{4}{3}$. This meets the real axis at $-\frac{1}{3}$. We therefore apply the isometry $z \mapsto \frac{z+\frac{1}{3}}{-\frac{1}{3}z-1} = \frac{3z+1}{-z-3}$. The composite of all these isometries is $z \mapsto \frac{3\left(\frac{3-4i}{5}\frac{z-0.5i}{-0.5iz-1}\right)+1}{-\left(\frac{3-4i}{5}\frac{z-0.5i}{-0.5iz-1}\right)-3} = \frac{3(3-4i)(z-0.5i)+(-0.5iz-1)}{-(3-4i)(z-0.5i)-3(-0.5iz-1)} = \frac{(9-12.5i)z-(7+4.5i)}{(-3+5.5i)z+5+1.5i}$.

12 *Find the endpoints of the hyperbolic line from $0.5-0.5i$ to $\frac{1-5i}{13}$ in the disc model. (i.e. find the points where this hyperbolic line meets the boundary of the disc.)*

First we apply the isometry $z \mapsto \frac{z-0.5+0.5i}{(0.5+0.5i)z-1}$. This sends $0.5-0.5i$ to the origin, and $\frac{1-5i}{13}$ to $\frac{2-10i-13+13i}{-4-6i-26} = \frac{-11+3i}{-30-6i} = \frac{1}{6}\frac{(-11+3i)(-5+i)}{5^2+1^2} = \frac{1}{6}\frac{52-26i}{26} = \frac{2-i}{6}$.

It therefore sends the hyperbolic line through $0.5-0.5i$ and $\frac{1-5i}{13}$ to the hyperbolic line from 0 to $\frac{2-i}{6}$, which meets the boundary of the unit disc at $\frac{2-i}{\sqrt{5}}$ and $\frac{-2+i}{\sqrt{5}}$. To find the endpoints of the original line, we need to apply the isometry $z \mapsto \frac{z-0.5+0.5i}{(0.5+0.5i)z-1}$ to these points. The first one goes to:

$$\frac{4-2i-\sqrt{5}+\sqrt{5}i}{3+i-2\sqrt{5}} = \frac{(4-\sqrt{5}+(\sqrt{5}-2)i)(3-2\sqrt{5}-i)}{30-12\sqrt{5}} =$$
$$\frac{20-10\sqrt{5}+(8\sqrt{5}-20)i}{30-12\sqrt{5}} = \frac{10-5\sqrt{5}+(4\sqrt{5}-10)i}{15-6\sqrt{5}}$$

[We can simplify this further to get $\frac{-\sqrt{5}-2i}{3}$.]

The second one goes to:

$$\frac{-4+2i-sqrt5+\sqrt{5}i}{-3-i-2\sqrt{5}} = \frac{(-4-\sqrt{5}+(2+\sqrt{5})i)(-3-2\sqrt{5}+i)}{30+12\sqrt{5}} =$$
$$\frac{20+10\sqrt{5}+(-20-8\sqrt{5})i}{30+12\sqrt{5}} = \frac{10+5\sqrt{5}-(10+4\sqrt{5})i}{3(5+2\sqrt{5})} =$$
$$\frac{\sqrt{5}-2i}{3}$$

13 *Describe the construction to trisect an acute angle using a straight-edge with a fixed distance marked on it and a pair of compasses, and prove that it does indeed produce an angle one third the size of the original.*

Let the angle be subtended at a point O by points A and B. Let the length marked on the straight-edge be l. Use the compasses to draw a circle of radius l about O, and let A' and B' be the points where this circle meets OA and OB respectively (on the same side of O as A and B). Extend OB past O. Move the straight-edge so that it passes through A', and so that the distance between the point C where it meets the line OB extended past O, and the point D where it meets the circle centre O, radius l be l. Now we know that $\triangle CDO$ is isosceles, so $\angle OCD = \angle DOC$. Also, $\triangle DOB'$ is isosceles, so $\angle ODB' = \angle OB'D$. By angles in a triangle, $\angle ODB' = \angle DOC + \angle DCO = 2\angle DCO$ and $\angle B'OA' = \angle OB'C + \angle OCB' = 3\angle OCB'$. Therefore $\angle OCB'$ is one third the size of the original angle AOB.