MATH 3090, Advanced Calculus I
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Homework Sheet 1
Model Solutions

Compulsory questions

1 *Prove from the definition of convergence that the sequence* $1, 2, 3, \ldots$ *does not converge to any real number* x.

We need to show that for any x,

$$(\exists \epsilon > 0)(\forall N)(\exists n \geqslant N)(|a_n - x| \geqslant \epsilon)$$

This means we can choose the ϵ. In this case any $\epsilon > 0$ works. We will take $\epsilon = 1$. Now there is some natural number $k > x$. If $n \geqslant k + 1$, then $|a_n - x| \geqslant |a_n - k| \geqslant 1$. So for any N, we can take $n = N + k + 1$. Then $n \geqslant N$, and $|a_n - x| \geqslant \epsilon$.

2 *(a) Show that if* (x_n) *is a sequence, such that every subsequence* (x_{n_i}) *has a subsequence which converges to* x, *then* $x_n \to x$. *[Hint: Suppose* x_n *does not converge to* x. *Then there is some* $\epsilon > 0$ *such that for every* N, *there is* $n > N$ *with* $|x_n - x| > \epsilon$. *Construct a sequence of these* x_n. *does it have a subsequence which converges to* x*?]*

Suppose x_n does not converge to x. Then there is some $\epsilon > 0$ such that for every N, there is $n > N$ with $|x_n - x| > \epsilon$. Choose n_0 so that $|x_{n_0} - x| > \epsilon$. Choose $n_1 \geqslant n_0 + 1$ so that $|x_{n_1} - x| > \epsilon$. Continue this process to get a subsequence $x_{n_0}, x_{n_1}, x_{n_2}, \ldots$ where each x_{n_i} satisfies $|x_{n_i} - x| \geqslant \epsilon$. Any subsequence of the x_{n_i} cannot converge to x, since it has no N such that for all $k \geqslant N$, $|x_{n_{i_k}} - x| < \epsilon$. However, this contradicts our initial assumption that any subsequence of x_n has a subsequence that converges to x. Therefore our supposition that x_n does not converge to x must be impossible, i.e. x_n must converge to x.

(b) Deduce that if y_n *is a bounded sequence that does not converge, then it has (at least) two convergent subsequences which converge to different limits. [Hint: If* x_n *does not converge to* x, *then as in part (a), we can construct a subsequence that has no subsequence converging to* x. *Use Bolzano-Weierstrass on this subsequence.]*

y_n has a convergent subsequence by the Bolzano-Weierstrass Theorem. Let y_{n_i} be a convergent subsequence, and let its limit be x. y_n does not converge to x, since it does not converge. Therefore, it cannot be the case that every subsequence y_{m_i} has a subsequence that converges to x, since by (a), this would force y_n to converge to x. Pick a subsequence y_{m_i} that has no subsequence converging to x. y_{m_i} is a bounded sequence (it has the same bounds as y_n) so by the Bolzano-Weierstrass theorem, it has a convergent subsequence $y_{m_{i_j}}$. The limit of $y_{m_{i_j}}$ cannot be x, so it must be some $y \neq x$. But $y_{m_{i_j}}$ is a subsequence of y_n that converges to y, and we already found a subsequence converging to x.

3 *Which of the following series converge and which diverge? Justify your answers. (You may assume convergence and divergence of the series covered in lectures.)*

(a) $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

ratio test:

If $a_n = \frac{3^n}{n!}$, then $\frac{a_{n+1}}{a_n} = \frac{n!3^{n+1}}{(n+1)!3^n} = \frac{3}{n+1} \to 0 as n \to \infty$

Therefore, by the ratio test, $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges.

root test:

$n! = \big((1 \times n) \times (2 \times (n-1)) \times \ldots \times (\frac{n}{2} \times \frac{n+2}{2})\big)$ (If n is odd, the last term in the product is just $\frac{n+1}{2}$). Each term in the product is at least n (except the term $\frac{n+1}{2}$ for n odd) so $n! \geqslant n^{\frac{n}{2}}$. Therefore, $\left(\frac{3^n}{n!}\right)^{\frac{1}{n}} \leqslant \frac{3}{\sqrt{n}} \to 0$, so by the root test, $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges.

(b) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

comparison test:

For $n \geqslant 2$, $\frac{n!}{n^n} = \frac{1 \times 2 \times \ldots}{n \times n \times \ldots} \leqslant \frac{2}{n^2}$, so $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

ratio test:

If $a_n = \frac{n!}{n^n}$, then $\frac{a_{n+1}}{a_n} = \frac{(n+1)!n^n}{n!(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n$. Now, $\left(\frac{n+1}{n}\right)^n = \left(1 + \frac{1}{n}\right)^n = \left(1 + \frac{n}{n} + \frac{n(n-1)}{2n^2} + \frac{n(n-1)(n-2)}{3!n^3} + \ldots + \frac{n!}{n!n^n}\right)$ As $n \to \infty$, the first few terms of the expansion tend to $1 + 1 + \frac{1}{2} + \frac{1}{3!} + \ldots$, and the last terms are very small, so the limit of $\left(1 + \frac{1}{n}\right)^n$ is e. Therefore, $\left(\frac{n}{n+1}\right)^n \to \frac{1}{e}$ as $n \to \infty$. As $\frac{1}{e} < 1$, $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges.

root test:

As above, $n! = \big((1 \times n) \times (2 \times (n-1)) \times \ldots \times (\frac{n}{2} \times \frac{n+2}{2})\big)$. All terms are at most $\left(\frac{n+1}{2}\right)^2$, so $(n!)^{\frac{1}{n}} \leqslant \frac{n+1}{2}$. Therefore $\left(\frac{n!}{n^n}\right)^{\frac{1}{n}} \leqslant \frac{n+1}{2n} \to \frac{1}{2} < 1$, so $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ converges by the root test.

(c) $\sum_{n=1}^{\infty} \sqrt{n^2+1} - n$ *[Hint:* $x^2 - y^2 = (x+y)(x-y)$*]*

comparison test:

$(\sqrt{n^2+1}-n)(\sqrt{n^2+1}+n) = (n^2+1-n^2) = 1$. Therefore, $\sqrt{n^2+1}-n = \frac{1}{\sqrt{n^2+1}+n}$, but $\sqrt{n^2+1}+n \leqslant 3n$ for $n \geqslant 1$, so $\sqrt{n^2+1}-n \geqslant \frac{1}{3n}$, so $\sum_{n=1}^{\infty} \sqrt{n^2+1} - n$ diverges by comparison to $\sum_{n=1}^{\infty} \frac{1}{3n}$.

integral test:

If $f(x) = \sqrt{x^2+1} - x$, then $f'(x) = \frac{2x}{2\sqrt{x^2+1}} - 1 < 0$ for $x > 0$, so f is a decreasing function of x, so the integral test can be applied.

Making the substitution $x = \sinh y$, we have $\int_0^N \sqrt{x^2+1}dx = \int_0^{\sinh^{-1} N} \cosh^2 y dy$. Using the identity $\cosh^2 y = \frac{1+\cosh(2y)}{2}$, this is $\int_0^{\sinh^{-1} N} \frac{1+\cosh(2y)}{2} dy = \left[\frac{y}{2} + \frac{\sinh(2y)}{4}\right]_0^{\sinh^{-1} N}$. Using $\sinh(2y) = 2\sinh(y)\cosh(y)$, this is $\frac{\sinh^{-1} N + N\sqrt{N^2+1}}{2}$. Therefore, $\int_0^N (\sqrt{x^2+1} - x)dx \geqslant \frac{\sinh^{-1} N}{2}$ which tends to ∞, so by the integral test, $\sum_{n=1}^{\infty} \sqrt{n^2+1} - n$ diverges.

(d) $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ *[Hint: to integrate* $\frac{1}{x \log x}$*, you may find the substitution* $u = \log x$ *helpful.]*

integral test:

Note that $f(x) = x \log x$ is an increasing function of x, so $g(x) = \frac{1}{x \log x}$ is a decreasing function of x. Therefore, $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ converges if and only if $\int_2^N \frac{1}{x \log x} dx$ converges as $n \to \infty$.

Let $u = logx$. $\frac{du}{dx} = \frac{1}{x}$. Therefore, $\int_2^N \frac{1}{x \log x} dx = \int_{\log 2}^{\log N} \frac{1}{ue^u}(e^u)du = [\log u]_{\log 2}^{\log N}$. But $\log(\log N) \to \infty$ as $N \to \infty$, so $\sum_{n=2}^{\infty} \frac{1}{n \log n}$ diverges.

comparison test:

The terms from $n = 2^k + 1$ to $n = 2^{k+1}$ are all at least $\frac{1}{2^{k+1}(k+1)\log 2}$, and there are 2^k values of n between $2^k + 1$ and 2^{k+1} inclusive. Therefore, $\sum_{n=2^k+1}^{2^{k+1}} \frac{1}{n \log n} \geqslant \frac{2^k}{2^{k+1}(k+1)\log 2} = \frac{1}{(2\log 2)(k+1)}$. Thus $\sum_{n=2}^{2^{k+1}} \frac{1}{n \log n} \geqslant \sum_{m=1}^{k} \frac{1}{(2\log 2)(m+1)}$, which diverges as $k \to \infty$. Therefore, $\sum_{n=2}^{2^{k+1}} \frac{1}{n \log n}$ also diverges.