MATH 3090, Advanced Calculus I
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Toby Kenney
Homework Sheet 2
Model Solutions

1 *Which of these series converge? In each case, determine whether the convergence is absolute. Justify your answers.*

(a) $\sum_{n=0}^{\infty}(-1)^n\frac{n^2+3}{n^3-7n+4}$

$(n^2+3)((n+1)^3-7(n+1)+4) = n^5+3n^4-n^3+7n^2-12n-6.$

$((n+1)^2+3)(n^3-7n+4) = n^5+2n^4-3n^3-10n^2-20n+16.$

Now for $n \geqslant 1$, $n^4+2n^3+17n^2+8n-10 > 0$, so $(n^2+3)((n+1)^3-7(n+1)+4) \geqslant ((n+1)^2+3)(n^3-7n+4)$. Therefore, $\frac{n^2+3}{n^3-7n+4} \geqslant \frac{(n+1)^2+3}{(n+1)^3-7(n+1)+4}$, so the terms in the series are decreasing in modulus. Also, for $n \geqslant 4$, $n^2 < n^2+3 < 2n^2$ and $n^3 > n^3-7n+4 > \frac{1}{2}n^3$, so $\frac{1}{n} < \frac{n^2+3}{n^3-7n+4} < \frac{4}{n}$, so the terms $\frac{n^2+3}{n^3-7n+4}$ tend to 0 as $n \to \infty$. Therefore, by the alternating series test, $\sum_{n=0}^{\infty}(-1)^n\frac{n^2+3}{n^3-7n+4}$ converges. On the other hand the series does not converge absolutely by comparison with $\frac{1}{n}$.

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\arctan n}$

As $n \to \infty$, $\arctan n \to \frac{\pi}{2}$, so $\frac{(-1)^n}{\arctan n}$ does not tend to zero. Therefore, $\sum_{n=1}^{\infty}\frac{(-1)^n}{\arctan n}$ diverges.

(c) $\sum_{n=1}^{\infty}(-1)^n\left(1-cos\left(\frac{1}{n}\right)\right)$

A good approximation to $\cos\theta$ is the following:

Claim: $\cos\theta \geqslant 1-\frac{\theta^2}{2}$. (note that these are the first two terms of the Taylor series of $\cos\theta$.

Proof of Claim: As cos is even, we only need to show the result for $\theta > 0$. In this case, $sin\theta \leqslant \theta$ since its derivative is $\leqslant 1$. Therefore, $\cos\theta = \cos(0) + \int_0^{\theta}(-\sin\phi)d\phi \geqslant 1 + \int_0^{\theta}(-\phi)d\phi = 1-\frac{\theta^2}{2}$.

An easier approximation to prove is that $\cos\theta \geqslant 1-\theta^2$, since $\cos^2\theta = 1-\sin^2\theta$, $(1-\cos\theta)(1+cos\theta) \leqslant \sin^2\theta$, so for $0 \leqslant \theta \leqslant \frac{\pi}{2}$, $1-\cos\theta \leqslant \sin^2\theta \leqslant \theta^2$. Therefore $\cos\theta \geqslant 1-\theta^2$. For $\theta \geqslant \frac{pi}{2}$, $1-\theta^2 < -1 \leqslant \cos\theta$.

Using either approximation, $1-cos\left(\frac{1}{n}\right) \leqslant \frac{1}{n^2}$, so $\sum_{n=1}^{\infty}(-1)^n\Big(1-cos\big(\frac{1}{n}\big)\Big)$ converges absolutely by comparison to $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

(d) $\sum_{n=0}^{\infty}(-1)^n\frac{2+(-1)^n}{n}$

$\sum_{n=0}^{\infty}(-1)^n\frac{2+(-1)^n}{n} = \sum_{n=0}^{\infty}\frac{2(-1)^n}{n} + \frac{(-1)^2n}{n} = \sum_{n=0}^{\infty}\frac{2(-1)^n}{n} + \frac{1}{n}$, and the first term converges by the alternating series test, while the second term diverges, so the whole series diverges. This shows that the condition in the alternating series test that the terms should be decreasing is necessary.

Alternatively, the sum of the nth and $n{+}1$th terms is $(-1)^n\left(\frac{2}{n(n+1)} + (-1)^n\left(\frac{1}{n}+\frac{1}{n+1}\right)\right)$, which is positive, and at least $\frac{1}{2n}$, so the series diverges by comparison to $\sum_{n=0}^{\infty}\frac{1}{2n}$.

(e) $\sum_{n=1}^{\infty}a_n$ *where* $a_n = \begin{cases} \frac{2}{m} & \textit{if } n = m^2 \\ \frac{-1}{n} & \textit{if } n \textit{ is not a perfect square} \end{cases}$

Consider the sum $\sum_{n=m^2+1}^{(m+1)^2}a_n$. Every term a_n for $n \neq (m+1)^2$ is $> \frac{-1}{m^2}$, and $< \frac{-1}{(m+1)^2}$. There are $2m$ such terms, so their sum S satisfies $\frac{-2m}{m^2} < S < \frac{-2m}{(m+1)^2}$. Therefore, $\frac{2}{m+1} - \frac{2m}{m^2} < \sum_{n=m^2+1}^{(m+1)^2}a_n < \frac{2}{m+1} - \frac{2m}{(m+1)^2}$. The lower bound is $\frac{-2}{m(m+1)}$, while the upper bound is $\frac{2}{(m+1)^2}$. Both $\sum_{n=0}^{\infty}\frac{-2}{m(m+1)}$ and $\sum_{n=0}^{\infty}\frac{2}{(m+1)^2}$ converge, so the sequence of partial sums $\sum_{n=1}^{m^2}a_n$ converges. Also, if $m^2 < N < (m+1)^2$, then $\left|\left(\sum_{n=1}^{N}a_n\right) - \left(\sum_{n=1}^{m^2}a_n\right)\right| < \frac{2}{m}$, and $\frac{2}{m} \to 0$ as $n \to \infty$, so $\sum_{n=1}^{\infty}a_n$ converges. $\sum_{n=1}^{\infty}a_n$ does not converge absolutely by comparison to $\sum_{n=1}^{\infty}\frac{1}{n}$.

2 *We showed (Theorem 6.18) that if* $\sum_{n=0}^{\infty}a_n$ *converges conditionally then* $\sum_{n=0}^{\infty}a_n^+$ *and* $\sum_{n=0}^{\infty}a_n^-$ *both diverge. Show that if* $\sum_{n=0}^{\infty}a_n$ *and* $\sum_{n=0}^{\infty}a_n^-$ *both diverge, but the sequence* $a_n \to 0$ *as* $n \to \infty$ *then there is a (conditionally) convergent rearrangement of* $\sum_{n=0}^{\infty}a_n$.

This is identical to the proof of 6.20 – the only facts we needed about a_n were that $\sum_{n=0}^{\infty}a_n^+$ and $\sum_{n=0}^{\infty}a_n^-$ both diverge, and that $a_n \to 0$ as $n \to \infty$.

3 *(a) Suppose* $\sum_{n=0}^{\infty}a_n$ *converges to* x, *and furthermore, suppose that the partial sums* $S_k = \sum_{n=0}^{k}a_n$ *are such that* $\sum_{n=0}^{\infty}\left|x - S_k\right|$ *converges. Prove that* $\sum_{n=0}^{\infty}a_n$ *converges absolutely. [Hint: use the triangle inequality* ($|a+b| \leqslant |a| + |b|$) *and the comparison test.]*

$a_k = S_k - S_{k-1}$ (where $S_{-1} = 0$). Therefore, $a_k = (x - S_{k-1}) - (x - S_k)$, so by the triangle inequality, $|a_k| \leqslant |x - S_{k-1}| + |x - S_k|$ ($|S_k - x| = |x - S_k|$). But as $\sum_{n=0}^{\infty} \left|x - S_k\right|$ and $\sum_{n=1}^{\infty} \left|x - S_{k-1}\right|$ are both convergent, so is $\sum_{n=1}^{\infty} \left|x - S_k\right| + \left|x - S_{k-1}\right|$, and therefore, by the comparison test, $\sum_{n=0}^{\infty} |a_n|$ is convergent, i.e. $\sum_{n=0}^{\infty} a_n$ is absolutely convergent.

(b) If $\sum_{n=0}^{\infty} \left|x - S_k\right|$ diverges, (Where, as in (a), $\sum_{n=0}^{\infty} a_n = x$ and $S_k = \sum_{n=0}^{k} a_n$ must the convergence of $\sum_{n=0}^{\infty} a_n$ be conditional? Give a proof or a counterexample.

$a_n = \frac{1}{n\sqrt{n}}$ is a counterexample. It converges absolutely to some value x, but $x - S_k$ is at least $\int_k^{\infty} x^{-\frac{3}{2}} dx = \left[-2x^{-\frac{1}{2}}\right]_k^{\infty} = \frac{1}{\sqrt{k}}$, by the integral test, so the sum $\sum_{n=0}^{\infty} \left|x - S_k\right|$ diverges.