MATH 3090, Advanced Calculus I
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Homework Sheet 3
Model solutions

1 *Define the sequence a_n recursively by $a_0 = 1$, and $a_n = \sum_{i=1}^{n} \frac{2a_{n-i}}{(i+2)!}$ for $n \geqslant 1$. Given that $\sum_{n=0}^{\infty} a_n$ converges, show that $\sum_{n=0}^{\infty} a_n = \frac{1}{2(3-e)}$. [Hint: Take the Cauchy product with the series $\sum_{n=0}^{\infty} \frac{1}{(n+2)!}$. Now use the relation $a_n = \sum_{i=1}^{n} \frac{2a_{n-i}}{(i+2)!}$ to simplify. The result should look similar to $\sum_{n=0}^{\infty} a_n$, and enable you to calculate it.]*

Observe that $a_{n+1} = \frac{a_n}{3} + \sum_{i=1}^{n} \frac{2a_{n-i}}{(i+3)!} \leqslant \frac{a_n}{3} + \sum_{i=1}^{n} \frac{2a_{n-i}}{3(i+2)!} = \frac{2a_n}{3}$, so $\sum_{n=0}^{\infty} a_n$ converges by comparison to a geometric series.
For $n \geqslant 1$, the nth term in the Cauchy product of $\sum_{i=0}^{\infty} \frac{1}{(i+2)!}$ and $\sum_{j=0}^{\infty} a_j$ is $\sum_{i=0}^{n} \frac{a_{n-i}}{(i+2)!} = \frac{a_n}{2} + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{2a_{n-i}}{(i+2)!}\right) = \frac{a_n}{2} + \frac{a_n}{2} = a_n$. For $n = 0$, the term is $\frac{a_0}{2}$, so the Cauchy product is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n$, so if $S = \sum_{n=0}^{\infty} a_n$, then $S(e-2) = S - \frac{1}{2}$, and thus $S = \frac{1}{2(3-e)}$.

2 *For each of the following functions, calculate the pointwise limit, f, if it exists, and determine whether the convergence is uniform. If no domain is specified, the f_n are functions on the whole of $\mathbb{R}$.*

(a) $f_n(x) = \begin{cases} 1 & \text{if } x < 0 \\ 1 - nx & \text{if } 0 \leqslant x \leqslant \frac{1}{n} \\ 0 & \text{if } x > \frac{1}{n} \end{cases}$

For $x \leqslant 0$, $f_n(x) = 1$ for all n, while for $x > 0$, if $n > \frac{1}{x}$, then $x > \frac{1}{n}$, so $f_n(x) = 0$. Therefore, the pointwise limit is

$$f(x) = \begin{cases} 1 & \text{if } x \leqslant 0 \\ 0 & \text{if } x > 0 \end{cases}$$

This is not a uniform limit, since for any n, $f_n\left(\frac{1}{2n}\right) = \frac{1}{2}$, so there is an x at which f_n is more than $\varepsilon = \frac{1}{4}$ from f.

(b) $f_n(x) = x^n e^{-nx^2}$

The pointwise limit is 0, since for any n and any $x > 0$, choose a so that $x = e^a$; now $f_n(x) = e^{n(a-e^{2a})}$; however, $e^{2a} > a$, so $f_n(x) \to 0$. For $x = 0$, $f_n(x) = 0$ for every n, and for $x < 0$, f_n is an odd function if n is odd, and an even function if n is even, so in either case, $f_n(x) \to 0$.

To see whether convergence is uniform, we find the maximum and minimum values of f_n on $\mathbb{R}$. $f_n'(x) = x^{n-1}e^{-nx^2} - 2nx^{n+1}e^{-nx^2}$. This is 0 when $x = 0$ (for $n > 1$) or when $2nx^2 = 1$. In the latter case, $f_n(x) = \pm\left(\frac{1}{2n}\right)^{\frac{n}{2}} e^{-\frac{1}{2}}$, which clearly tends to 0 as $n \to \infty$. Therefore, $f_n \to 0$ uniformly.

(c) $f_n(x) = \sin\left(\frac{x}{n}\right)$

As $\frac{x}{n} \to 0$ for all x and sine is a continuous function, the pointwise limit is the constant $\sin 0 = 0$. The convergence is not uniform, since for any n, $f_n\left(\frac{\pi n}{2}\right) = 1$.

(d) $f_n(x) = \sin(nx)$

This does not have a pointwise limit, since for example, if $x = \frac{\pi}{2}$

$$f_n(x) = \begin{cases} 0 & \text{if } x \text{ is even} \\ (-1)^{\frac{x-1}{2}} & \text{if } x \text{ is odd} \end{cases}$$

which does not converge.

(e) $f_n(x) = x^n$ *for* x *in the interval* $(0, 1)$ *(endpoints not included).*

The pointwise limit is 0. The convergence is not uniform, as $f_n\left(\left(\frac{1}{2}\right)^{\frac{1}{n}}\right) = \frac{1}{2}$, which does not tend to 0.

(f) $f_n(x) = \begin{cases} \frac{1}{n} & \textit{if } x = \frac{p}{n^q} \textit{for integers } p \textit{ and } q \\ 0 & \textit{otherwise} \end{cases}$

For every x, $|f_n(x)| < \frac{2}{n}$, so $f_n \to 0$ uniformly as $n \to \infty$.

3 *Let* f_n *be a sequence of continuous functions converging uniformly to* f *(which is therefore continous). Suppose that* $x_n \to x$ *is a sequence of real numbers. Show that* $f_n(x_n) \to f(x)$*. (You may assume that if* f *is continuous and* $a_n \to a$*, then* $f(a_n) \to f(a)$*.) [Hint: for* $\varepsilon > 0$*, first choose* N *so that for* $n > N$*,* $|f(x_n) - f(x)| < \frac{\varepsilon}{2}$*, then choose* $M > N$ *so that* $|f_M - f| < \frac{\varepsilon}{2}$*. Do not choose* M *before* N *– it won't work!]*

For any $\varepsilon > 0$, we choose N so that for every $n \geqslant N$, $|f(x) - f(x_n)| < \frac{\varepsilon}{2}$. Now we choose M such that for any $m \geqslant M$ and any y, $|f_m(y) - f(y)| < \frac{\varepsilon}{2}$. Now we have for $n \geqslant N + M$,

$$|f_n(x_n) - f(x)| \leqslant |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore, $f_n(x_n) \to f(x)$ as $n \to \infty$.

To see why choosing M first doesn't work, observe that if we say

$$|f_n(x_n) - f(x)| \leqslant |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)|$$

then we need to be able to pick N so that for every $n \geqslant M$, and every $m \geqslant N$, $|f_n(x_m) - f_n(x)| < \frac{\varepsilon}{2}$. This means that we need for every x and every $\varepsilon > 0$, a $\delta > 0$ which demonstrates that all of the f_n are continuous. This is an important property that a sequence of functions might have, but it is not implied by uniform convergence.