MATH 3090, Advanced Calculus I
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Homework Sheet 4
Model solutions

Compulsory questions

1 *Which of the following series of functions converge uniformly on the interval (0,1)? If they do not converge uniformly, is the limit continuous?*

(a) $\sum_{n=0}^{\infty}\frac{x^n}{x+n}$ *[You may assume that* $\left(1-\frac{1}{N}\right)^N \geqslant \frac{1}{12}$ *for* $N \geqslant 2$*.]*

This does not converge uniformly, since $\sum_{n=N}^{\infty}\frac{x^n}{x+n} \geqslant \sum_{n=N}^{\infty}\frac{x^n}{n+1}$, and for any $N \geqslant 2$, if we let $x = 1-\frac{1}{N}$, then $x^N \geqslant \frac{1}{12}$ (see below) and for $N \leqslant n < 2N$, $x^n \geqslant \frac{1}{144}$, so $\frac{x^n}{n+1} \geqslant \frac{1}{288N}$. There are N terms between N and $2N$, so their sum is at least $\frac{1}{288}$. Therefore the series does not converge uniformly.

The limit is continuous, because the convergence is uniform on the interval $(0,R]$ for any $R<1$, as $\frac{x^n}{x+n}$ is an increasing function of x in the interval $(0,1)$ for any $n \geqslant 1$ (its derivative is $\frac{nx^{n-1}(x+n)-x^n}{(x+n)^2}$, which is positive in $(0,1)$). The terms of the series (after the $n=0$ term) are therefore bounded by $\frac{R^n}{R+n}$, so convergence is uniform on the interval $(0,R)$ by the Weierstrass M-test with $M_n = \frac{R^n}{R+n}$.

To show that $\left(1-\frac{1}{N}\right)^N \geqslant \frac{1}{12}$, *for* $N \geqslant 2$, *we note that its binomial expansion is an alternating series begining* $1-1+\frac{N-1}{2N}-\frac{(N-1)(N-2)}{6N^2}+\ldots$. *The terms are decreasing in modulus, so by the alternating series test, the sum is at least* $1-1+\frac{N-1}{2N}-\frac{(N-1)(N-2)}{6N^2}$, *which is at least* $1-1+\frac{1}{4}-\frac{1}{6}=\frac{1}{12}$.

In fact, $\left(1-\frac{1}{N}\right)^N \to e^{-1}$, *this can be seen by observing that its binomial expansion is approximately the power series for* e^x *evaluated at* -1. *We can show that the difference between its binomial expansion and the power series for* e^{-1} *tends to 0 as* $n \to \infty$.

(b) $\sum_{n=1}^{\infty}\frac{1}{n^2+x}$

As $x>0$, $\frac{1}{n^2+x} < \frac{1}{n^2}$, so the series converges uniformly by the Weierstrass M-test, with $M_n = \frac{1}{n^2}$.

(c) $\sum_{n=1}^{\infty}\frac{\cos(n(x+1))}{n}$ *[Hint: multiply by* $2\sin\left(\frac{x+1}{2}\right)$. *Recall that* $2\sin\alpha\cos\beta =$

$\sin(\beta+\alpha) - \sin(\beta-\alpha)$. *There should then be cancellation between consecutive terms of the resulting series.]*

$$2\sin\left(\frac{x+1}{2}\right)\sum_{n=1}^{\infty}\frac{\cos(n(x+1))}{n} = \sum_{n=1}^{\infty}\frac{2\sin\left(\frac{x+1}{2}\right)\cos(n(x+1))}{n}$$
$$= \sum_{n=1}^{\infty}\frac{\sin\left(\left(n+\frac{1}{2}\right)(x+1)\right) - \sin\left(\left(n-\frac{1}{2}\right)(x+1)\right)}{n}$$

But the $\sin\left(\left(n+\frac{1}{2}\right)(x+1)\right)$ and the $\sin\left(\left((n+1)-\frac{1}{2}\right)(x+1)\right)$ terms partially cancel, to give

$$\left(\sum_{n=1}^{\infty}\frac{\sin\left(\left(n+\frac{1}{2}\right)(x+1)\right)}{n(n+1)}\right) - \sin\left(\frac{1}{2}(x+1)\right)$$

which converges uniformly by the Weierstrass M-test, where $M_n = \frac{1}{n(n+1)}$.

Now, for $0 < x < 1$, $0.1 < 2\sin\left(\frac{x+1}{2}\right) < 2$, so for any $x \in (0,1)$,

$$\sum_{n=N}^{\infty}\frac{\cos(n(x+1))}{n} < 10\left(2\sin\left(\frac{x+1}{2}\right)\right)\left(\sum_{n=N}^{\infty}\frac{\cos(n(x+1))}{n}\right)$$

Therefore, $\sum_{n=1}^{\infty}\frac{\cos(n(x+1))}{n}$ also converges uniformly.

2 *(a)Suppose (f_n) is a sequence of continuously differentiable functions on an interval $[a,b]$, converging pointwise to f. Suppose the derivatives f_n' converge uniformly to g on $[a,b]$. (In Theorem 7.12 we showed that g is the derivative of f.) Prove that $f_n \to f$ uniformly on $[a,b]$. (You may assume that $\left|\int_x^y f(t)dt\right| \leqslant \int_x^y |f(t)|dt$.)*

On $[a,b]$, $f(x) = f(a) + \int_a^x g(t)dt$, while $f_n(x) = f_n(a) + \int_a^x f_n'(t)dt$. Given $\epsilon > 0$, we can choose N and M so that $|f(a) - f_n(a)| < \frac{\epsilon}{2}$ for all $n \geqslant N$, and $|g(t) - f_m'(t)| < \frac{\epsilon}{2(b-a)}$ for all $m \geqslant M$ and all $t \in [a,b]$. Now

$$|f(x) - f_n(x)| = \left|f(a) - f_n(a) + \int_a^x g(t) - f_n'(t)dt\right|$$

$$\leqslant |f(a) - f_n(a)| + \int_a^x |g(t) - f_n'(t)|dt < \frac{\epsilon}{2} + \frac{\epsilon(x-a)}{2(b-a)} < \epsilon$$

Therefore, $f_n \to x$ uniformly on $[a,b]$.

(b) What if instead of the finite interval $[a,b]$, the sequence f_n converges pointwise to f on the interval $[a,\infty)$, and $f_n' \to g$ uniformly on $[a,\infty)$?

Now the argument above won't work because our integral might be arbitrarily long. Consider $f_n(x) = \frac{x}{n}$, and $f(x) = 0$. We have that $f_n \to 0$ pointwise on $(-\infty, \infty)$, and $f'_n \to 0$ uniformly on $(-\infty, \infty)$, but f_n does not converge uniformly on any interval $[a, \infty)$.

3 *Find the radius of convergence of each of the following power series. Do they converge at the points where $|x|$ is equal to the radius of convergence?*

(a) $\sum_{n=0}^{\infty} \frac{x^n}{n^3+2n+3}$

As $n \to \infty$, $\frac{(n+1)^3+2(n+1)+3}{n^3+2n+3} \to 1$. Therefore, by the ratio test, if $|x| < 1$ then $\sum_{n=0}^{\infty} \frac{x^n}{n^3+2n+3}$ converges, while if $|x| > 1$, $\sum_{n=0}^{\infty} \frac{x^n}{n^3+2n+3}$ diverges. Therefore, the radius of convergence is 1.

When $|x| = 1$, $\sum_{n=0}^{\infty} \frac{x^n}{n^3+2n+3}$ converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^3}$.

(b) $\sum_{n=0}^{\infty} \frac{x^{\left(n^2\right)}}{n!}$

$\frac{x^{\left((n+1)^2\right)}}{x^{\left(n^2\right)}} = x^{2n+1}$, while $\frac{(n+1)!}{n!} = n+1$. Therefore, the ratio of consecutive terms in the series is $\frac{x^{2n+1}}{n+1}$, which tends to zero if $|x| < 1$, and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

When $|x| = 1$, the series converges (e.g. by the ratio test).

(c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{2^n(n+3)}$

The ratio test tells us that this series converges if $\frac{x^2(n+4)}{2(n+3)}$ tends to a limit that is < 1, and diverges if it tends to a limit that is > 1. However, the limit is $\frac{x^2}{2}$, so the radius of convergence is $\sqrt{2}$.

When $x = \pm\sqrt{2}$, the series is $\sum_{n=0}^{\infty} \frac{1}{n+3}$, which diverges by comparison to $\sum_{n=0}^{\infty} \frac{1}{2n}$.