MATH 3090, Advanced Calculus I
Fall 2006
Toby Kenney
Homework Sheet 5
Model Solutions

Compulsory questions

1 *(a) Find the radius of convergence of* $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^{n+1}}$.

The ratio between consecutive terms is $\frac{-x^2}{4}$, so the radius of convergence is 2 by the ratio test.

(b) Evaluate $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{4^{n+1}}$ *on the interval* $(-R,R)$, *where* R *is the radius of convergence [Hint: it's a geometric series]. On what interval is the function you get infinitely differentiable?*

The sum is a geometric series with common ratio $\frac{-x^2}{4}$, so its sum is $\frac{1}{4}\left(\frac{1}{1+\frac{x^2}{4}}\right) = \frac{1}{x^2+4}$ on (-2,2). The function $f(x) = \frac{1}{n^2+4}$ is infinitely differentiable on the whole of $\mathbb{R}$.

When we study complex numbers, we will see why the Taylor expansion of $\frac{1}{n^2+4}$ *only has radius of convergence 2.*

2 *Let*

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Differentiate $f(x)$. *Show that* $x^n f(x) \to 0$ *as* $x \to 0$, *for any* n. *Can* f *be expressed as a Taylor series about 0?*

$f'(x) = \frac{-2e^{-\frac{1}{x^2}}}{x^3}$. To show that $x^{-n}f(x) \to 0$ as $x \to 0$, we show that given any n, for sufficiently large x, $e^x > x^n$.

To do this we observe first that $e^x \geqslant ex$ by seeing that they are equal when x=1, and that the derivative of e^x is more than e when $x > 1$ and less than e when $x < 1$. Now this means that for any m, $e^x = e^{m\frac{x}{m}} = \left(e^{\frac{x}{m}}\right)^m \geqslant \left(\frac{ex}{m}\right)^m$. Now if $m = n+1$, then when $x > \frac{m^m}{e^m}$, we have that $e^x \geqslant \left(\frac{xe^m}{m^m}\right)x^n > x^n$.

Now, since $\frac{x^{-(n+1)}}{x^{-n}} \to 0$ as $x \to \infty$, we have that $x^n e^{-x} \to 0$ as $x \to \infty$. Therefore, $x^n e^{-x^2} \to 0$ as $x \to \infty$, so $x^{-n}f(x) \to 0$ as $x \to 0$, since $x^{-1} \to \infty$ as $x \to 0$.

By the product rule, every derivative of f on $\mathbb{R} \setminus \{0\}$ is the product of a polynomial in x^{-1} multiplied by $f(x)$. Therefore, $\frac{\frac{d^n f}{dx^n}}{x} \to 0$ as $x \to 0$, so $\left.\frac{d^{(n+1)} f}{dx^{n+1}}\right|_0 = 0$ for all n. Therefore, f does not have a Taylor expansion about 0, since all the terms in it would be 0.

3 *Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence R, and suppose $x_0 \in (-R, R)$. Show that $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a Taylor series expansion about x_0 with radius of convergence at least $R - |x_0|$. [Hint: Calculate the coefficients as power series in x_0 by differentiating the series repeatedly. Now observe that $\sum_{n=0}^{\infty} |a_n| \left(\sum_{m=0}^{n} \binom{n}{m} |x_0|^{n-m} |x - x_0|^m \right)$ converges when $|x - x_0| < R - |x_0|$. Therefore, we can rearrange the terms to get that $\sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} |a_n| \binom{n}{m} |x_0|^{n-m} |x - x_0|^m \right)$ converges. Compare this to the Taylor series we got by differentiating at x_0.]*

We know that the mth derivative of f at x_0 is the sum $\sum_{n=m}^{\infty} \frac{n!}{(n-m)!} a_n x_0^{n-m}$. Therefore, the Taylor series expansion of f about x_0 is

$$\sum_{m=0}^{\infty} \left(\sum_{n=m}^{\infty} \binom{n}{m} a_n x_0^{n-m} (x - x_0)^m \right)$$

We also know that $\sum_{m=0}^{n} \binom{n}{m} |x_0|^{n-m} |x - x_0|^m = (|x_0| + |x - x_0|)^n$, so for $|x_0| + |x - x_0| < R$, the series

$$\sum_{n=0}^{\infty} a_n \sum_{m=0}^{n} \binom{n}{m} x_0^{n-m} (x - x_0)^m$$

is an absolutely convergent double series. Therefore, we can rearrange its terms without affecting the result. In particular,

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} a_n \binom{n}{m} (x_0)^{n-m} (x - x_0)^m$$

is absolutely convergent. But this is the Taylor series above.

4 *Find power series about 0 for the following integrals:*
(a) $\int_{t=0}^{x} \cos(t^3) dt$

$\cos(t^3)$ has power series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n}}{(2n)!}$. The integral of this is the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+1}}{(6n+1)(2n)!}$. There is no constant term because the integral starts at 0, so the value at $x = 0$ is 0.

$\int_{t=0}^{x} \frac{e^t-1}{t} dt$

The power series for $\frac{e^t-1}{t}$ is $\sum_{n=0}^{\infty} \frac{t^n}{(n+1)!}$. Therefore, when we integrate, we get $\sum_{n=1}^{\infty} \frac{t^{n+1}}{(n+1)(n+1)!}$. Again, there is no x^0 term because we are integrating from 0.