MATH 3090, Advanced Calculus I
Fall 2006
Toby Kenney
Homework Sheet 7
Model Soultions

1 *Find the Fourier coefficients for the following functions. You may use either* $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ *or* $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$.

(a) $f(x) = \begin{cases} -1 & \text{if } (2n-1)\pi < x \leqslant 2n\pi \\ 1 & \text{if } 2n\pi < x \leqslant (2n+1)\pi \end{cases}$ *for any integer* n.

f is an odd function, so the a_n will all be 0. The b_n are given by

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{\pi}\int_0^{\pi} \sin nx dx - \frac{1}{\pi}\int_{-\pi}^{0} \sin nx dx = \frac{2}{\pi}\int_0^{\pi} \sin nx dx$$
$$= \begin{cases} \frac{4}{n\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

Therefore, the Fourier series is $\sum_{n=0}^{\infty} \frac{4\sin(2n+1)x}{2n+1}$.

(b) $f(x) = y^3$ *where* y *is the value of* $x - 2n\pi$ *with smallest modulus, and* $f((2n+1)\pi) = 0$ *(so* $f(x) = x^3$ *for* $-\pi < x < \pi$*, and* f *is* 2π*-periodic).*

f is an odd function, so all the a_n are 0. We calculate the b_n as follows:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^3 \sin nx dx = \frac{1}{\pi}\left(\left[\frac{-x^3\cos nx}{n}\right]_{-\pi}^{\pi} + \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{3x^2\cos nx}{n}dx\right)$$
$$= (-1)^{n+1}\frac{2\pi^2}{n} + \frac{1}{\pi}\left[\frac{3x^2\sin nx}{n^2}\right]_{-\pi}^{\pi} - \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{6x\sin nx}{n^2}dx$$
$$= (-1)^{n+1}\frac{2\pi^2}{n} + \frac{1}{\pi}\left[\frac{6x\cos nx}{n^3}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{6\cos nx}{n^3}dx = \frac{(-1)^{n+1}2\pi^2}{n} + \frac{(-1)^n 12}{n^3}$$

So $f(x) = \sum_{n=1}^{\infty}(-1)^n \frac{12-2\pi^2 n^2}{n^3}\sin nx dx$.

2 *(a) Show that* $f(x)$ *given by* $f(x) = x^4$ *on* $[-\pi, \pi)$*, and* f *is* 2π*-periodic, has Fourier series* $f(x) = \frac{\pi^4}{5} + \sum_{n=1}^{\infty}(-1)^n 8\left(\frac{\pi^2}{n^2} - \frac{6}{n^4}\right)\cos nx$.

f is an even function, so the b_n are all 0. We calculate the a_n by integrating by parts. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{2\pi^4}{5}$.

$$\int_{-\pi}^{\pi} x^4 \cos nx = \left[\frac{x^4 \sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{4x^3\sin nx}{n}dx$$

but $\sin n\pi = 0$, so the first term is 0, leaving

$$-\int_{-\pi}^{\pi} \frac{4x^3 \sin nx}{n} dx = \left[\frac{4x^3 \cos nx}{n^2}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{12x^2 \cos nx}{n^2} dx$$

Here the first term is $(-1)^n \frac{8\pi^3}{n^2}$, and

$$\int_{-\pi}^{\pi} \frac{12x^2 \cos nx}{n^2} dx = \left[\frac{12x^2 \sin nx}{n^3}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{24x \sin nx}{n^3} dx$$

Again, the first term is 0, so we just need to evaluate

$$\int_{-\pi}^{\pi} \frac{24x \sin nx}{n^3} dx = \left[\frac{24x \cos nx}{n^4}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{24 \sin nx}{n^4} dx$$

We know the integral is 0, while the first term is $\frac{(-1)^n 48\pi}{n^4}$. Therefore, $f(x) = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} (-1)^n 8 \left(\frac{\pi}{n^2} - \frac{6}{n^4}\right) \cos nx$.

(b) Deduce that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} = -\frac{7\pi^4}{720}$ *(You may assume that* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$*).*

Comparing $f(x)$ with its Fourier series when $x = 0$, we see that $0 = \frac{\pi^4}{5} + \sum_{n=1}^{\infty} (-1)^n 8 \left(\frac{\pi^2}{n^2} - \frac{6}{n^4}\right) \cos 0$. Therefore, $\sum_{n=1}^{\infty} (-1)^n \frac{\pi^2}{n^2} - \sum_{n=1}^{\infty} (-1)^n \frac{6}{n^4} = -\frac{\pi^4}{40}$. However, we know that $\sum_{n=1}^{\infty} (-1)^n \frac{\pi^2}{n^2} = -\frac{\pi^4}{12}$, so we have $\sum_{n=1}^{\infty} (-1)^n \frac{6}{n^4} = \frac{\pi^4}{40} - \frac{\pi^4}{12} = -\frac{7\pi^4}{120}$, and $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^4} = -\frac{7\pi^4}{720}$.

3 *Suppose* $f(x)$ *is a* 2π*-periodic function with Fourier series* $f(x) = \frac{1}{2}a_0 + \sum_{n=0}^{\infty} a_n \cos((4n+1)x)$*. Express the function* $g(x)$ *whose Fourier series is* $g(x) = \frac{1}{2}a_0 \cos x - \frac{1}{2}a_0 \sin x + \sum_{n=0}^{\infty} a_n \cos((4n+2)x)$ *in terms of* $f(x)$*. [Hint:* $\cos((4n+2)x) = \cos x \cos((4n+1)x) - \sin x \sin((4n+1)x)$*, and* $\sin((4n+1)x) = \cos(4n+1)\left(x - \frac{\pi}{2}\right)$*.]*

We know that $\cos((4n+2)x) = \cos x \cos((4n+1)x) - \sin x \sin(4n+1)x$, and that $\sin(4n+1)x = \cos(4n+1)\left(x - \frac{\pi}{2}\right)$. Therefore,

$$g(x) = \tfrac{1}{2}a_0 \cos x - \tfrac{1}{2}a_0 \sin x + \sum_{n=0}^{\infty} \cos x \cos(4n+1)x - \sum_{n=0}^{\infty} \sin x \cos(4n+1)\left(x - \tfrac{\pi}{2}\right) = f(x) \cos x - f\left(x - \tfrac{\pi}{2}\right) \sin x$$