MATH 3090, Advanced Calculus I
Fall 2006
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Homework Sheet 8
Model Solutions

1 *Recall that the Fourier series for $f(x) = x$ when $-\pi \leqslant x < \pi$, and f 2π-periodic is $2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin nx}{n}$. By integrating this 4 times, find the Fourier series for $g(x) = \frac{x^5}{120} - \frac{\pi^2 x^3}{36} + \frac{7\pi^4 x}{360}$. [Remember to add the constant terms.]*

$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{x^2}{2}dx = \frac{\pi^2}{6}$, so the Fourier series for $\frac{x^2}{2} - \frac{\pi^2}{6}$ is the integral of the Fourier series for x, so it is $2\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$. By integrating again, $\frac{x^3}{6} - \frac{\pi^2 x}{6} = 2\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n^3}$. Now $\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{x^4}{24} - \frac{\pi^2 x^2}{12}\right)dx = \frac{-7\pi^4}{360}$, so by integrating again, we get $\frac{x^4}{24} - \frac{\pi^2 x^2}{12} + \frac{7\pi^4}{360} = 2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\cos nx}{n^4}$. By integrating once more, we get $\frac{x^5}{120} - \frac{\pi^2 x^3}{36} + \frac{7\pi^4 x}{360} = 2\sum_{n=0}^{\infty}(-1)^{n+1}\frac{\sin nx}{n^5}$.

2 *Find the Fourier sine and cosine series for the following functions on the interval $[0, \pi]$.*

(a) $f(x) = e^x$ [Hint: $\cos nx = \frac{e^{inx} + e^{-inx}}{2}$, $\sin nx = \frac{e^{inx} - e^{-inx}}{2i}$.]

$$\begin{aligned}
\int_0^{\pi} e^x \sin nx dx = \int_0^{\pi}\frac{e^{(1+ni)x} - e^{(1-ni)x}}{2i}dx &= \frac{1}{2i}\left(\left[\frac{e^{(1+ni)x}}{1+ni}\right]_0^{\pi} - \left[\frac{e^{(1-ni)x}}{1-ni}\right]_0^{\pi}\right)\\
&= \frac{1}{2i(1+n^2)}\left((1-ni)\left(e^{\pi}e^{ni\pi} - 1\right) - (1+ni)\left(e^{\pi}e^{-ni\pi}\right)\right)\\
&= \frac{1}{2i(1+n^2)}\left(2ni + e^{\pi}\left((1-ni)e^{ni\pi} - (1+ni)e^{-ni\pi}\right)\right)\\
&= \frac{n}{1+n^2} + e^{\pi}\left(\frac{\left(e^{ni\pi} - e^{-ni\pi}\right) - ni\left(e^{ni\pi} + e^{-ni\pi}\right)}{2i(1+n^2)}\right)\\
&= \frac{n + e^{\pi}\left(\sin n\pi - n\cos n\pi\right)}{1+n^2} = \frac{n\left(1 + (-1)^{n+1}e^{\pi}\right)}{1+n^2}
\end{aligned}$$

Therefore, the Fourier sine series for e^x on $[0, \pi]$ is $e^x = \sum_{n=1}^{\infty}\frac{2n\left(1+(-1)^{n+1}e^{\pi}\right)}{\pi(1+n^2)}\sin nx$.

Similarly,

$$\int_0^{\pi} e^x \cos nx dx = \int_0^{\pi}\frac{e^{(1+ni)x} + e^{(1-ni)x}}{2}dx = \frac{1}{2}\left(\left[\frac{e^{(1+ni)x}}{1+ni}\right]_0^{\pi} + \left[\frac{e^{(1}{1}\right.\right.$$

$$= \frac{1}{2(1+n^2)}\left((1-ni)\left(e^{\pi}e^{ni\pi}-1\right)+(1+ni)\left(e^{\pi}e^{-\right.\right.$$

$$= \frac{1}{2(1+n^2)}\left(e^{\pi}\left((1-ni)e^{ni\pi}+(1+ni)e^{-\right.\right.$$

$$= e^{\pi}\left(\frac{\left(e^{ni\pi}+e^{-ni\pi}\right)-ni\left(e^{-ni\pi}-e^{-ni\pi}\right)}{2(1+n^2)}\right)-\frac{1}{1+n^2}=e^{\pi}\left(\frac{\cos n\pi+n\sin(-n\pi)}{1+n^2}\right)-\frac{1}{1+n^2}=\frac{(-}{}$$

Therefore, the Fourier cosine series for e^x on $[0,\pi]$ is $e^x = \frac{e^\pi - 1}{\pi} + \sum_{n=1}^{\infty} \frac{2((-1)^n e^\pi - 1)}{\pi(1+n^2)} \cos nx$.

(b) $f(x) = \sin\left(x + \frac{\pi}{3}\right)$.

$\sin\left(x+\frac{\pi}{3}\right) = \sin x \cos\frac{\pi}{3} + \cos x \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x$. Now for $n \geqslant 2$,

$$\int_0^{\pi} \sin x \cos nx dx = \int_0^{\pi} \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right) dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{n+1} - \frac{1}{n-1} & \text{if } n \text{ is even} \end{cases}$$

and

$$\int_0^{\pi} \cos x \sin nx dx = \int_0^{\pi} \frac{1}{2}\left(\sin(n+1)x + \sin(n-1)x\right) dx = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{1}{n+1} + \frac{1}{n-1} & \text{if } n \text{ is even} \end{cases}$$

On the other hand, $\int_0^{\pi} \sin x \sin nx dx = 0$ when $n \neq 1$, and $\int_0^{\pi} \cos x \cos nx = 0$ when $x \neq 1$.

Therefore, $f(x) = \frac{\sqrt{3}}{2}\sin x + \sum_{n=1}^{\infty}\left(\frac{1}{2n+1} + \frac{1}{2n-1}\right)\frac{2\sin 2nx}{\pi}$, is the sine series for f, and $f(x) = \frac{1}{2}\cos x + \sum_{n=1}^{\infty}\left(\frac{1}{2n+1} - \frac{1}{2n-1}\right)\frac{\sqrt{3}\cos 2nx}{\pi}$ is the cosine series.

3 *Define f by $f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$. (This converges for all x by Dirichlet's test – see Corollary 6.27.)*

(a) Show that the series converges uniformly on the intervals (δ, π) and $(-\pi, -\delta)$ for any $\delta > 0$. (This means that f is a continuous function everywhere except perhaps at integer multiples of π.)

Examining the proof of Dirichlet's test (Theorem 6.25), we see that the rate of convergence depends only on the rate of convergence of the sequence a_n, and on the bound C. Therefore, If we let $a_n = \frac{1}{n}$, and $b_n(x) = \sin nx$, then by Lemma 6.26,

$$\left|\sum_{n=0}^{N} b_n(x)\right| = \left|\frac{\sin\frac{(k+1)x}{2}\sin\frac{kx}{2}}{\sin\frac{x}{2}}\right| \leqslant \frac{1}{\sin\frac{x}{2}} \leqslant \frac{1}{\sin\frac{\delta}{2}}$$

Therefore, the convergence is uniform on (δ, π) by Dirichlet's test.

By subtracting off a multiple of the square wave ($h(x) = \begin{cases} -1 & \text{if } (2n-1)\pi < x \leqslant 2n\pi \\ 1 & \text{if } 2n\pi < x \leqslant (2n+1)\pi \end{cases}$) *and a multiple of the sawtooth wave* ($s(x) = x$ *for* $-\pi < x \leqslant \pi$, *and* 2π-*periodic) from* f, *we get a function* g *that is continuous at all* x.

(b) Show that g *is not piecewise continuously differentiable. [Hint: if it were piecewise continuously differentiable, what would the Fourier coefficients have to be? Use Bessel's inequality to show that these cannot be the Fourier coefficients of a piecewise continuous function.]*

If f were piecewise continuously differentiable, then its derivative f' would have Fourier coefficients $a'_n = nb_n = \sqrt{n}$ so g' will have Fourier coefficients $\sqrt{n} + \alpha_n$ $(n \geqslant 1)$ where α_n is the term that comes from the multiple of the square wave and the sawtooth wave that we subtracted from f. α_n is bounded for all n, since the Fourier coefficients of the sawtooth and the square waves decay like $\frac{1}{n}$. However, these a'_n do not satisfy Bessel's inequality ($\sum_{n=0}^{\infty} |c'_n|^2 \leqslant \int_{-\pi}^{\pi} |f'(x)|^2 dx$). Therefore, they are not the Fourier coefficients of a piecewise continuous 2π-periodic function.