MATH 3090, Advanced Calculus I
Fall 2006
Toby Kenney
Homework Sheet 9
Model Solutions

Compulsory questions

1 *An rod of length π and 0 thickness is heated to a uniform $100^\circ C$ at time 0. The ends of the rod are then immersed in ice, to fix their temperature at $0^\circ C$.*

(a) When the temperature in the middle of the rod ($x = \frac{\pi}{2}$) reaches $\frac{400}{\pi} \times \left(\frac{1}{2} - \frac{1}{3\times 2^9}\right)^\circ C$, show that the temperature at the point one third of the way along the rod ($x = \frac{\pi}{3}$) is at most $\frac{100\sqrt{3}}{\pi}^\circ C$. [Hint: Both the temperature in the middle of the rod, and the temperature one third of the way along the rod can be expressed as alternating series (you have to combine some terms) with terms decreasing in modulus. Recall that if a series is alternating and its terms have decreasing modulus, then the partial sums give alternately lower and upper bounds for the whole sum. You should be able to show that $e^{-kt} \leqslant \frac{1}{2}$.]

The temperature satisfies the heat equation $u_t = -ku_{xx}$ and the boundary conditions $u(0,t) = u(\pi,t) = 0$, and the initial condition $u(0,x) = 100$. The solution is therefore

$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-kn^2 t} \sin(nt)$$

where the a_n are the Fourier coefficients of the sine series for the constant function 100. Therefore, $a_n = \frac{2}{\pi}\int_0^\pi 100 \sin nx dx = \begin{cases} \frac{400}{\pi n} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$,

so the temperature in the middle of the rod at time t is $\frac{400}{\pi}\sum_{n=0}^{\infty} \frac{(-1)^n e^{-(2n+1)^2 kt}}{2n+1}$. This is an alternating series whose terms have decreasing modulus, so it is at least $\frac{400}{\pi}\left(e^{-kt} - \frac{(e^{-kt})^9}{3}\right)$, so when it reaches $\frac{400}{\pi} \times \left(\frac{1}{2} - \frac{1}{3\times 2^9}\right)$, we must have that $e^{-kt} \leqslant \frac{1}{2}$. (Since if $f(x) = x - \frac{x^9}{3}$, then we have that $f(e^{-kt}) \leqslant f\left(\frac{1}{2}\right)$, and f has only one local maximum in $[0,1]$, and it is more than $\frac{1}{2}$, and $f(1) > f\left(\frac{1}{2}\right)$, so if $f(x) \leqslant f\left(\frac{1}{2}\right)$, then we must have $x \leqslant \frac{1}{2}$.)

Similarly, the temperature at the point one third of the way along the rod at time t is $\frac{400}{\pi}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)^2kt}\sin(2n+1)x}{(2n+1)}$, but

$$\sin(2n+1)\frac{\pi}{3} = \begin{cases} \frac{\sqrt{3}}{2} & \text{if } n = 3m \\ 0 & \text{if } n = 3m+1 \\ -\frac{\sqrt{3}}{2} & \text{if } n = 3m+2 \end{cases}$$

so the temperature is $\frac{400\sqrt{3}}{2\pi}\sum_{m=0}^{\infty}\left(\frac{e^{-(6m+1)kt}}{6m+1} - \frac{e^{-(6m+5)kt}}{6m+5}\right)$. This is an alternating series, so it is at most the first term, which is $\frac{400\sqrt{3}}{2\pi} \times e^{-kt} \leqslant \frac{100\sqrt{3}}{\pi}$.

(b) Show that once the $e^{-4kt} \leqslant \frac{1}{2}$, the temperature never gets below $0°C$ anywhere on the rod. (This is true for all positive time, but it is easier to show if we make the assumption that $e^{-4kt} \leqslant \frac{1}{2}$.) [Hint: show that by this time, the first term of the Fourier series solution is larger in modulus than the sum of the moduli of all the others. This term is positive for all $x \neq 0, \pi$, so the temperature must be non-negative for all $x \neq 0, \pi$. You may assume that $\left|\frac{\sin kx}{\sin x}\right| \leqslant k$ for all integers k.]

The temperature $u(x,t)$ is given by $u(x,t) = \frac{400}{\pi}\sum_{n=0}^{\infty}\frac{e^{-(2n+1)^2kt}\sin(2n+1)x}{2n+1}$, so the first term is $e^{-kt}\sin x$, and for the subsequent terms: $|e^{-(2n+1)^2kt}\sin(2n+1)x| \leqslant e^{-(2n+1)^2kt}(2n+1)\sin x$. Therefore, the sum of terms after the first has modulus at most

$$e^{-kt}\sin x\sum_{n=1}^{\infty}e^{-4n^2kt} \leqslant e^{-kt}\sin x\sum_{n=1}^{\infty}e^{-4nkt} = \left(e^{-kt}\sin x\right)\frac{e^{-4kt}}{1-e^{-4kt}}$$

Since $e^{-4kt} \leqslant \frac{1}{2}$, we have that $\frac{e^{-4kt}}{1-e^{-4kt}} \leqslant 1$, so the first term is at least as large as the sum of the others, and the first term is non-negative for all x, so the sum can never be negative.

2 *A guitar string is plucked by pulling the part of the string with x coordinate a a distance 1 upwards. The equation for the string is therefore*

$$f(x) = \begin{cases} \frac{x}{a} & \text{if } 0 \leqslant x \leqslant a \\ \frac{\pi-x}{\pi-a} & \text{if } a \leqslant x \leqslant \pi \end{cases}$$

where $f(x)$ is the vertical displacement of the string at point x. It is released from rest in this position at time 0 (so $u(x,0) = f(x)$, $u_t(x,0) = 0$).

(a) Assuming that the Fourier series solution (8.37) does indeed model the future behaviour of the string, calculate the Fourier coefficients b_n of $\cos nt\sin nx$ as functions of a.

$b_n = \frac{2}{\pi}\left(\int_0^a \frac{x \sin nx}{a} dx + \int_a^\pi \frac{(\pi - x)\sin nx}{\pi - a} dx\right)$. By the change of variable $y = \pi = x$ in the second integral, this becomes $\frac{2}{\pi}\left(\int_0^a \frac{x \sin nx}{a} dx + \int_0^{\pi - a} \frac{(-1)^n y \sin ny}{\pi - a} dx\right)$. But $\int_0^a \frac{x \sin nx}{a} dx = [\frac{-x \cos nx}{na}]_0^a + \int_0^a \frac{\cos nx}{na} dx = \frac{\sin na}{n^2 a} - \frac{\cos na}{n}$. Therefore, $b_n = \frac{2}{\pi}\left(\frac{\sin na}{n^2 a} - \frac{\cos na}{n} + \frac{\sin n(\pi - a)}{n^2(\pi - a)} - \frac{\cos n(\pi - a)}{n}\right)$. However, $\sin(\pi - a) = \sin a$ and $\cos(\pi - a) = -\cos a$, so this expression simplifies to $b_n = \frac{2 \sin na}{\pi n^2}\left(\frac{1}{a} + \frac{1}{\pi - a}\right) = \frac{2 \sin na}{n^2 a(\pi - a)}$.

(b) Show that $a = \frac{\pi}{2}$ gives a local maximum of the coefficient b_1. We will see next week that (assuming that $\frac{\pi}{2}$ is the global maximum) this means that the middle of the string is the best place to pluck it if we want to get as close to a pure note as possible.

When $n = 1$, we have $b_1 = \frac{2 \sin a}{a(\pi - a)}$. Differentiating with respect to a, we get $b_1' = \frac{2a(\pi - a)\cos a - 2(\pi - 2a)\sin a}{a^2(\pi - a)^2}$. When $a = \frac{\pi}{2}$, $\cos a = 0$, and $\pi - 2a = 0$, so the numerator is 0, while the denominator is not 0, so $\frac{\pi}{2}$ is a critical point of b_1. To show it is a maximum, we take the second derivative: $b_1'' = \frac{2a^2(\pi - a)^2(-a(\pi - a)\sin a + (\pi - 2a)\cos a - (\pi - 2a)\cos a + 2 \sin a) + 4a(\pi - a)(\pi - 2a)(2a(\pi - a)\cos a - 2(\pi - 2a)\sin a)}{a^4(\pi - a)^4}$. When $a = \frac{\pi}{2}$, this becomes $\frac{\frac{\pi^2}{2}\left(-\frac{\pi^2}{4} + 2\right)}{\frac{\pi^8}{256}}$, which is negative. Therefore, $a = \frac{\pi}{2}$ is a local maximum for b_1.

3 *Suppose we modify the wave equation to account for the fact that the string is not perfectly elastic. We use the equation:*

$$u_{tt} = c^2 u_{xx} - 2\delta u_t$$

where δ is a small positive constant. Assume that $\delta < c$. Use separation of variables to find a family of solutions to this equation that satisfy the boundary conditions $u(0, t) = u(\pi, t) = 0$.

Suppose the solution is of the form $u(x, t) = \Theta(x)\Phi(t)$ for some function Θ and Φ. Now the equation becomes $\Theta(x)\ddot{\Phi}(t) = c^2\Theta''(x)\Phi(t) - 2\delta\Theta(x)\dot{\Phi}(t)$. When we divide through by $\Theta(x)\Phi(t)$, we get $\frac{\ddot{\Phi}(t) + 2\delta\dot{\Phi}(t)}{\Phi(t)} = \frac{c^2\Theta''(x)}{\Theta(x)}$. As in the case without damping, the left-hand side depends only on t, and the right-hand side depends only on x, so they must both be equal to some constant α. From the right-hand side, we get that $\alpha = -c^2 n^2$ for some $n \in \mathbb{Z}^+$. The left-hand side therefore gives the equation $\ddot{\Phi}(t) + 2\delta\dot{\Phi}(t) + c^2 n^2 \Phi(t) = 0$. We look for solutions to this of the form $\Phi(t) = e^{\lambda t}$ for some $\lambda \in \mathbb{C}$, and we get $e^{\lambda t}\left(\lambda^2 + 2\delta\lambda + c^2 n^2\right) = 0$. Therefore, we get that $\lambda = -\delta \pm \sqrt{\delta^2 - c^2 n^2}$. The solution is therefore $e^{-\delta t}\left(a_n \cos\left(\sqrt{c^2 n^2 - \delta^2}\right) t + b_n \sin\left(\sqrt{c^2 n^2 - \delta^2}\right) t\right) \sin nx$.