MATH 2112/CSCI 2112, Discrete Structures I
Winter 2007
Toby Kenney
Mock Final Examination
Time allowed: 3 hours
Calculators not permitted.

Answer all questions.

1 Use Euclid's algorithm to find the greatest common divisor of 263 and 184. Write down all the steps involved. Use your calculations to find integers a and b such that $263a + 184b = (263, 184)$ times the second number is their greatest common divisor.

2 Are the propositions $q \to (p \to r)$ and $p \to (q \to r)$ logically equivalent? Justify your answer.

3 Which of the following are true when $A = \{1, 2, 8\}$ and $B = \{0, 4, 5, 9, 17\}$? Justify your answers.

(a) $(\forall x \in B)(x + 1 \in B)$

(b) $(\exists x \in A)(x + 3 \in A)$

(c) $(\forall x \in A)(\exists y \in B)(\forall z \in A)(x + y + z \text{ is prime})$

4 Use universal instantiation and rules of inference to show that the following argument is valid:

$$(\forall x \in A)(\neg(x \in B))$$
$$(y \in A \vee y \in C) \wedge (y \in B \vee y \in C)$$
$$\therefore y \in C$$

5 Prove or disprove the following. You may use results proved in the course or the homework sheets, provided you state them clearly.

(a) There is a natural number n such that $n^2 + 5n - 6$ is prime.

(b) $2^{19} + 3^8 + 7^{84}$ is divisible by 5.

6 Find $0 \leqslant n < 770$ satisfying all the following congruences:

$$n \equiv 4 \pmod{11} \quad (1)$$
$$n \equiv 2 \pmod{14} \quad (2)$$
$$n \equiv 3 \pmod{5} \quad (3)$$

7 Solve the following recurrence relations:

(a) $a_n = a_{n-1} + 2a_{n-2}$, $a_0 = 1$, $a_1 = 3$.

(b) $a_n = 6a_{n-1} - 9a_{n-2}$, $a_0 = 0$, $a_1 = 4$.

(c) $a_n = 2a_{n-1} + 3$, $a_0 = 3$.

8 Show by induction on n that if A is a set of n elements, then its power set $\mathcal{P}(A)$ has 2^n elements. [Hint: let $a \in A$, and consider $\mathcal{P}(A)$ as the union of the set of subsets of A that contain a, and the set of subsets of A that don't contain a.]

9 Let $A = \{0, 1, 3, 7\}$, $B = \{1, 2, 7, 8\}$. What are:

(i) $A \cup B$?

(ii) $A \cap B$?

(iii) $A \times B$?

(iv) $B \setminus A$?

10 Let A, B, and C be sets such that $|A| = 7$, $|B| = 9$, $|C| = 17$, $|A \cap B| = 4$, $|A \cap C| = 3$, $|B \cap C| = 7$, and $|A \cup B \cup C| = 21$. What are the possible values for $|A \cap B \cap C|$?

11 For each of the following relations, determine which of the properties: reflexivity, symmetry, transitivity, and antisymmetry hold:

(a) The relation R on the set of all sets given by $A\ R\ B$ if and only if $\emptyset \in A \wedge \emptyset \in B$.

(b) The relation R on the set of natural numbers given by $n\ R\ m$ if and only if $n|m$.

(c) The relation R on the set of all natural numbers given by $m\ R\ n$ if and only if m is odd and n is even.

(d) The relation R on the set of rational numbers given by $q\ R\ r$ if $q = \frac{a}{b}$, $r = \frac{c}{d}$ with $(a, b) = (c, d) = 1$, $a, b, c, d \in \mathbb{Z}^+$ and $ad < b$.

12 (a) Which of the following functions are injective? (b) Which are surjective?

(i) $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$.

(ii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$.

(iii) $f : \mathbb{N} \to \{0, 1\}$, $f(n) = \begin{cases} 0 & \text{if } n \text{ is prime} \\ 1 & \text{otherwise} \end{cases}$.

(iv) $f : \mathbb{Q} \to \mathbb{R}$, $f(x) = 2x$.

13 Show by strong induction that every positive integer congruent to 2 modulo 3 is divisible by a (positive) prime number congruent to 2 modulo 3.

14 Show that it is not possible to write a computer program which takes as input a computer program P, and some value X, and determines whether the programs P eventually finishes when given input X.

15 Consider the following algorithm, called a bubble sort for sorting a list $a[1], a[2], \ldots, a[n]$ of length n.

Algorithm 1 Bubble Sort

```
Input: List a[1], a[2], ..., a[n]
Output: Sorted list a[1], a[2], ..., a[n]
  numSwaps=1
  while numSwaps>0 do
    numSwaps=0
    for i=1 to n-1 do
      Compare a[i] to a[i+1]
      if a[i] > a[i+1] then
        swap a[i] and a[i+1]
        numSwaps=numSwaps+1
      end if
    end for
  end while
```

How many comparisons does it make to sort a list of length n: (Give your answers in the form $\Theta(f(n))$ for some function f), justify your answers.

(a) In the best case?

(b) In the worst case? [Hint: Every time the outer loop runs, we know that for every $i < n$, there is at least one more $j > i$ with $a[j] > a[i]$.]

16 Define the function $F : \mathbb{N} \to \mathbb{N}$ recursively by:

$$F(n) = \begin{cases} 4F\left(\frac{n}{2}\right) & \text{if } n \text{ is even.} \\ F(n-1) + 2n - 1 & \text{if } n \text{ is odd} \end{cases}$$

and $F(0) = 0$.

Find a formula for $F(n)$, and prove it.

17 Given a set X of 10 natural numbers $\{n_1, \ldots, n_{10}\}$, for a non-empty subset X' of X, define $S_{X'} = \sum_{i \in X'} n_i$. show that there are two non-empty subsets X_0 and X_1 of X such that $S_{X_0} \equiv S_{X_1} \pmod{1000}$.