Bonus Question 3

An **integral** of a function $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$. We write:

$$F(x) = \int f(x)dx \qquad \Longleftrightarrow \qquad F'(x) = f(x)$$

For example, $\int x dx = x^2/2 + C$ and $\int \frac{1}{x} dx = \ln(x) + C$ where C is some arbitrary constant.

(a) Find constants A, B such that

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}.$$

(b) Using part (a), find $\int \frac{1}{x(x-1)} dx$.

(c) Often, a differential equation can be reduced to an integral. For example, note that

$$[\ln(y(x))]' = \frac{y'(x)}{y(x)}.$$

Therefore the solution to a differential equation $y' = ay$ can be obtained by rewriting it as

$$\frac{y'}{y} = a$$

and taking the integral of both sides to obtain

$$\begin{aligned} \ln(y) &= ax + C, \\ y &= \exp(ax)\exp(C). \end{aligned}$$

(d) A **logistic equation** of population growth is the differential equation

$$\frac{dy}{dx} = y(1-y).$$

Generalize part (c) to reduce this equation to an integral in part (b). Then use the result you found in part (b) to find a solution to the logistic equation, subject to an additional constraint,

$$y(0) = \frac{1}{2}.$$

What happens to $y(t)$ in the limits $t \to +\infty$ and $t \to -\infty$?