Bonus Question 4

In this question, you are asked to use calculus to prove the following classical identities:

$$a^{x+y} = a^x a^y; \tag{1}$$

$$\sin(x+y) = \sin(x)\cos(y) + \sin(y)\cos(x); \tag{2}$$

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y). \tag{3}$$

1. a) As was shown in bonus question 3, if $z(x)$ is a solution to the differential equation

$$z'(x) = z(x) \tag{4}$$

then $z(x)$ must be of the form

$$z(x) = Ce^x \tag{5}$$

for some constant C. Use this fact to prove (1) for the special case where $a = e$.

b) Use part (a) to show that (1) is true for any (positive) a.

2. a) Show that $z = \cos(x)$ and $z = \sin(x)$ are solutions to the differential equation

$$z''(x) = -z(x). \tag{6}$$

Then show that

$$z = A\cos(x) + B\sin(x) \tag{7}$$

is also a solution to (7), for any constants A and B.

b) In fact, it is also possible to show that the opposite is true: any solution of (6) must be of the form (7). [You are not asked to prove this fact here; it is proven in a second year differential equations class]. Use this fact to prove the identity (2) and (3).