Math 1000 Review questions

Final exam: 11 December, 8:30AM, in Dalplex. NO NOTES, NO CALCULATORS!!!

1. You need to know basic facts about these functions:

$$f(x) = \frac{1}{x},\ x^p,\ e^x,\ \sin x,\ \cos x,\ \tan x$$

and their inverses: $f(x) = \frac{1}{x},\ x^{1/p},\ \ln(x),\ \sin^{-1} x$, etc. The basic facts include:

- Domain and range; various asymptotes and graphs (example: what happens to $\ln x$ as $x \to 0$, or as $x \to \infty$?),
- Basic values (e.g. what is $\ln(1), \ln(e), \arctan(1)$)
- Their derivatives. For inverse trigs, you need to know how to find these.

2. Limits: compute

$$\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7};\quad \lim_{x\to\infty} \frac{2x+1}{\sqrt{x^2+\sin(x)}};\quad \lim_{x\to\infty} \left(\tan^{-1}(2x) - \frac{\pi}{2}\right) x;$$
$$\lim_{x\to 0} (x \cot(2x));\quad \lim_{x\to\infty} e^{-x} x^{2006};\quad \lim_{x\to 1^+} \ln(x-1).$$

3. Differentiation from first principles: $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Example: Find the derivative of $f(x) = \sqrt{1+2x}$ using the definition of derivative. No credit will be given if you use differentiation rules but you can use them to check your answer.

4. Basic differentiation for example $\frac{d}{dx}\tan(x\cos 2x)$, inverse trigs (example: $\frac{d}{dx}\tan^{-1}(x)$), log differentiation (example: $\frac{d}{dx}\left((\cos x)^{2x}\right)$).

5. Implicit functions: Suppose that $\tan y + \arctan x = y^2 x + 1$. Find $\frac{dy}{dx}$ at the point $x = 0,\ y = \frac{\pi}{4}$.

6. Linear approximation:

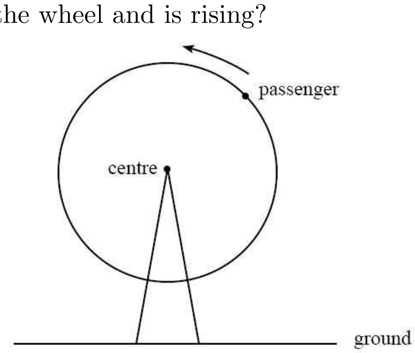
   (a) For question 5, estimate y when $x = 0.1$.

   (b) Estimate $\ln(3)$.

7. Related rates:

   (a) A man 2 m tall walks toward a lamppost on level ground at a rate of 0.5 m/s. If the lamp is 5 m high on the post, how fast is the length of the man's shadow decreasing when he is 3 m from the post? How fast is the shadow of his head moving at that time?

   (b) A circular ferris wheel with radius 10 metres is revolving at the rate of 10 radians per minute. How fast is a passenger on the wheel rising when the passenger is 6 metres higher than the centre of the wheel and is rising?



   (c) A lump of clay is being rolled out so that it maintains the shape of a circular cylinder (and its volume remains constant). If the length is increasing at a rate proportional to itself, show that the radius is decreasing at a rate proportional to itself. Note: $V = \pi r^2 l$.

8. Graphing: Sketch the graph of a given function $f(x)$. Indicate behaviour as $x \to \pm\infty$, any vertical asymptotes, roots, critical points, max/min, concavity. For example, $f(x) = xe^x$; $f(x) = \frac{x}{x^2-4}$; $f(x) = 2|x-1| + 5$.

9. More graphing: Suppose $f(x)$ is an even function ($f(x) = f(-x)$), continuous and differentiable everywhere, and has the following properties:

   - It has two inflection points for $x > 0$ at $x = 1$ and $x = 3$ and $f'(2) = 0$;
   - $f(x) > 0$ for all x and $f(x) \to 1$ as $x \to \infty$.

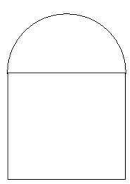
   Sketch the graph of f. Indicate any critical points, max/min, concavity. Does $f(x)$ have any roots?

10. Max/min:

    (a) Find the absolute minimum and the absolute maximum value of $f(x) = \frac{x}{x^2+1}$ on the interval $[-\frac{1}{2}, 3]$.

    (b) At noon, a sailboat is 20 km due south of a freighter. The sailboat is travelling due east at 20 km/hr, and the freighter is travelling due south at 40 km/hr. When are the two ships closest to each other, and what is the distance between them at that point?

    (c) A window has the shape of a rectangle surmounted by a semicircle (see diagram). For a fixed perimeter P, what should the dimensions of the rectangle and semicircle be, in order to maximize the area of such a window?



11. A particle is moving on a straight line and its acceleration at time t is given by $a(t) = \sqrt{t}$. At time $t = 0$ its position is given by $p(0) = 2$ and its velocity is $v(0) = 3$. Find the the position of the particle at time $t = 1$.

12. Set up the Riemann sum to approximate the area under the curve $y = \frac{1}{x^3+1}$ between $x = 1$ and $x = 2$ with $n = 3$ subintervals and with the sample points taken at the left endpoints. Sketch a diagram to illustrate.

13. Show that $\int_1^2 \dfrac{1}{x^3+1} dx \leq 3/8$. (hint: first show that $\dfrac{1}{x^3+1} \leq \dfrac{1}{x^3}$)

14. Evaluate the following:

$$\int \frac{x^2+\sqrt{x}}{x} dx; \quad \int_0^{\pi/2} \frac{\cos x}{1+\sin(x)} dx; \quad \int e^{2x} dx; \quad \int_0^1 \frac{1}{x^2+1} dx; \quad \int_0^4 \frac{(\sqrt{x}+1)^3}{\sqrt{x}} dx;$$

$$\int_{-2}^{2} \sin\left(x^3\right) dx.$$

15. Suppose that $g(x) = \int_1^{x^2} \dfrac{1}{\sqrt{t}+t^2} dt$. Find $g(1)$ and $g'(1)$.

16. IVT, MVT etc:

    (a) Using the Intermediate Value Theorem, show that there is at least one positive solution to $x^3 - 1 = x$. Then show that that there is exactly one such solution with $x > 0$ using either Rolle's or Mean Value Theorem.

    (b) A function $f(x)$ satisfies $f(0) = 0$, $f(1) = 2$ and $f(2) = -1$. It is known that f is differentiable everywhere. Show that $f'(c) = 0$ for some number c. Give complete justification, specifying any relevant theorems.

    (c) Show that $\sin(x) < x$ for all $x > 0$.