# Math 1500X, Final Exam (December 2017)

No calculator or cheat sheets are allowed. You have 3 hours. Please write all your answers in the booklet provided.

1. Determine the following limits.

2 (a) $\lim_{x\to\infty} \frac{(x-3)(2x+5)(3x+7)}{\sqrt{5x^6+7}}$ 2 (b) $\lim_{x\to 0} \frac{\sin(2\sin(x))}{\sin(3x)}$ 2 (c) $\lim_{t\to\infty} \frac{t^2-\sqrt{t^4+3t^3+5}}{t}$

2. 2 (a) State the delta-epsilon definition of a limit.

3 (b) Let $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \le 0. \end{cases}$ . Use the delta-epsilon definition of a limit to show that the limit $\lim_{x\to 0} f(x)$ does not exist.

2 3. Find the derivative of $f(x) = \sqrt{x}$ from the first principles (no credit given for using differentiation rules).

4. Find the derivative of the following functions using any method you like:

2 (a) $y = \ln(\ln x)$ (b) $y = (\ln(x))^{\sin(x)}$ 2

5. 2 (a) Find the limits $\lim_{x\to 0^+} x^{-x}$ and $\lim_{x\to\infty} x^{-x}$. 3 (b) Sketch the graph of $y = x^{-x}$, $x > 0$. Label any max/min or asymptotes.

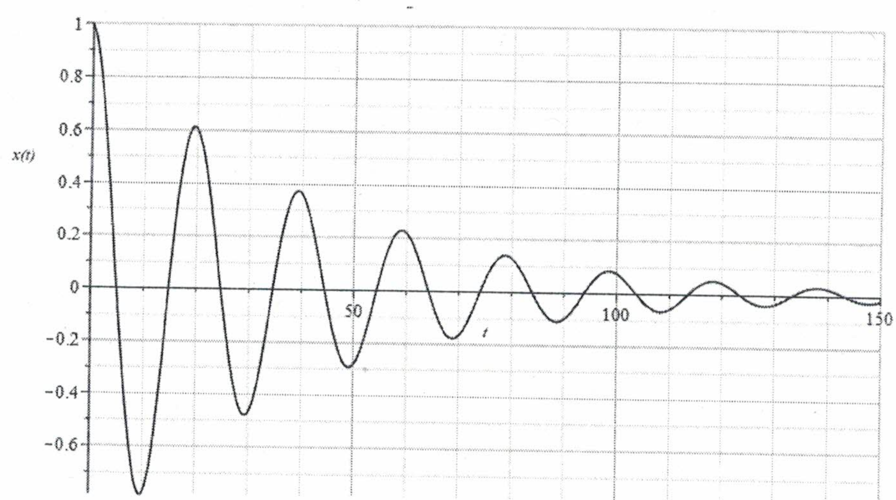
3 6. Show that the function $f(x) = 2x + \cos(x)$ has *exactly one* root, and give bounds for where the root is located.

7. 2 (a) Estimate $\ln(1.1)$ using linear approximation about an appropriate point. 2 (b) Estimate the error in part (a).

8. 5 Find the solution to the initial value problem

$$x'' - 4x' + 13x = 0, \quad x(0) = 1, \quad x'(0) = 0.$$

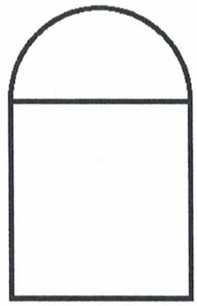
9. Recall that the motion of an object attached to a spring satisfies $mx'' + cx' + kx = 0$, where $x(t)$ is the displacement from the equilibrium, $m$ is its mass, $c$ is the friction coefficient and $k$ is the spring constant. For a certain spring, the displacement $x(t)$ is as shown in the graph below.




If the mass is known to be $m = 1kg$, estimate (a) the friction coefficient $c$, and (b) the spring constant $k$. You may express your answer in "calculator ready" format.

10. A Norman window has the shape of a rectangle surmounted by a semicircle, as shown below.

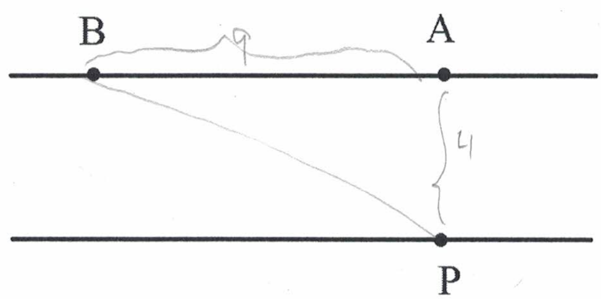
6



If the total perimeter of the window is 10m, find the dimensions of the window so that the greatest possible amount of light is admitted.

11. A person in a row boat at point P is 4km away from a straight shore line. The point A on the shore is directly opposite the boat. The objective is to travel from point P to point B on the shore a distance 9km from A in a minimum amount of time. If the person can row at 4 km per hour and walk at 5 km per hour, where should the person land the boat between A and B?

6




12. [BONUS QUESTION]

(a) Suppose that $f(0) = -1$ and $f''(x) > 0$ for all $x \geq 0$. Show that $f(x)$ has at most one positive root.

(b) Sketch an example of a function $f(x)$ with that satisfies the conditions of part (a) but has no positive roots.

(c) Suppose that $f(0) = -1$ and $f''(x) \geq 1$ for all $x \geq 0$. Show that $f(x)$ has exactly one positive root.

1)

a) $\sim \dfrac{6x^3}{\sqrt{5}\,x^3} \sim \boxed{\dfrac{6}{\sqrt{5}}}$ b) $\boxed{\dfrac{2}{3}}$ c)

c) $$\frac{t^2 - \sqrt{\phantom{x}}}{t} = \frac{(t^2 - \sqrt{\phantom{x}})(t^2 + \sqrt{\phantom{x}})}{t(\sqrt{\phantom{x}} + t^2)}$$

$\sim t^2$

$$\sim \frac{t^4 - (t^4 + 3t^3 + 5)}{t(t^2 + t^2)} \sim \boxed{-\frac{3}{2}} \text{ as } t \to \infty$$

2) (a) $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ s.t. $|f(x) - l| < \varepsilon$ whenever $0 < |x - a| < \delta$

(b) Choose $\varepsilon = \frac{1}{2}$.

- If $x > 0$ then $|f(x) - l|$ $= |1 - l| < \frac{1}{2} \Rightarrow l \in (0.5, 1.5)$

- If $x < 0$ then $|f(x) - l| =$ $= |0 - l| < \frac{1}{2} \Rightarrow l \in (-\frac{1}{2}, \frac{1}{2})$

So $l \in (0.5, 1.5) \cup (-\frac{1}{2}, \frac{1}{2}) = \emptyset$

No such $l$ exists.

3) $\dfrac{\sqrt{x+h}-\sqrt{x}}{h} = \dfrac{(\sqrt{x+h}-\sqrt{x})(\sqrt{x+h}+\sqrt{x})}{h(\sqrt{x+h}+\sqrt{x})} \sim \dfrac{h}{h\,2\sqrt{x}}$

$\Rightarrow \dfrac{1}{2\sqrt{x}}$ as $h \to 0$

4) (a) $\dfrac{1}{\ln x}\,\dfrac{1}{x}$

(b) $y = e^{\ln(\ln x^{\sin x})} = e^{(\sin x)(\ln \ln x)}$

$y' = e^{(\sin x)(\ln \ln x)} \cdot \left[\cos x \ln \ln x + \sin x \dfrac{1}{\ln x}\dfrac{1}{x}\right]$

5) $y = x^{-x} = e^{-x \ln x}$ Note that $x \ln x \to \infty$ as $x \to \infty$

$\to 0$ as $x \to 0^+$

So $y(x) \to 0$ as $x \to \infty$

$\to 1$ as $x \to 0^+$

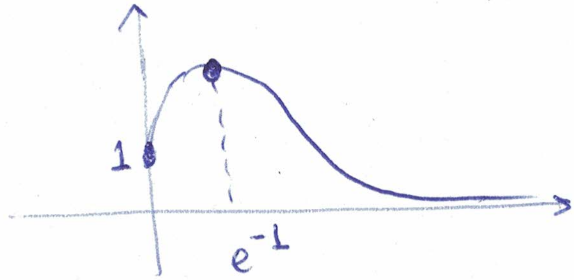
$y' = e^{-x\ln x}(-\ln x - 1) = 0 \Rightarrow \ln x = -1$

$x = e^{-1}$

$\Rightarrow$ <u>max</u> at $x = e^{-1}$.

[cannot be min since $y > 0$ and $y \to 0$ and unique]

Sketch:




$$y(e^{-1}) = e^{-e^{-1}\ln(e^{-1})}$$
$$= e^{e^{-1}}$$

- Horizontal asymptote at $x=\infty$
- Vertical tangent at $x=0$

6) $f(x) \to \pm\infty$ as $x \to \pm\infty$ so $\exists$ at least one root by IVT. Bounds: $f\left(\frac{\pi}{2}\right)=\pi$; $f\left(-\frac{\pi}{2}\right)=-\pi$

so $\exists$ root $\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Uniqueness: Note that $f' = 2-\sin x \geq 1 > 0$

so $f$ is increasing

$\Rightarrow$ root is unique.

7)
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2}f''(\xi), \quad \xi \in [a, a+h].$$

Apply with $a=1$, $h=0.1$, $f(x)=\ln x$. Then:

$$\ln(1.1) = \underbrace{\ln 1}_{0} + 0.1\,\frac{1}{1} + \frac{(0.1)^2}{2}\left(\frac{-1}{\xi^2}\right) \qquad f' = \frac{1}{x}, \quad f'' = -\frac{1}{x^2}$$

$$= 0.1 - \frac{0.005}{\xi^2}, \quad \xi \in [1, 1.1]$$

$\Rightarrow \ln(1.1) \approx 0.1$

$$\text{Error} = |\ln(1.1) - 0.1| = \frac{0.005}{\xi^2}, \quad \xi \in (1, 1.1)$$

$$\leq 0.005 \quad \left[\text{since } \frac{1}{\xi^2} \text{ is decr.}\right].$$

8) $x = e^{\lambda t} \Rightarrow$ (1) $\lambda^2 - 4\lambda + 13 = 0$

$$\lambda = \frac{4 \pm \sqrt{16 - 13 \cdot 4}}{2}$$

16+5 4·13

(1) $= 2 \pm 3i$

(1)

$\Rightarrow x = e^{2t}\left[A \sin 3t + B \cos 3t\right]$

i.c.

(2)

$$\left.\begin{array}{l} x'(0) = 2B + 3A = 0 \\ x(0) = B = 1 \end{array}\right\} \Rightarrow \begin{array}{l} A = -\frac{2}{3} \\ B = 1 \end{array}$$

$$x = e^{2t}\left[-\frac{2}{3}\sin(3t) + \cos(3t)\right]$$

9) $\lambda = -\frac{c}{2} \pm \sqrt{\frac{c^2}{4} - mk} = \underbrace{-\frac{c}{2}} \pm i \underbrace{\sqrt{k - \frac{c^2}{4}}}_{\omega}$

- Sol'n looks like

$$y = e^{-\frac{c}{2}t} \cos(\omega t).$$

- From graph, $y(40) \sim 0.4\, y(0)$

$$\Rightarrow \quad e^{-\frac{c}{2}\cdot 40} = 0.4$$

$$-c \cdot 20 = \ln 0.4$$

$$\boxed{c \approx -\frac{\ln 0.4}{20}}$$

- Frequency: two full periods between $t = [0, 40]$

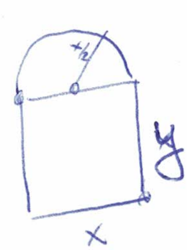
$$\Rightarrow \quad \omega \cdot 40 = 2 \cdot (2\pi)$$

$$\Rightarrow \omega \approx \frac{\pi}{10}$$

$$k - \frac{c^2}{4} \approx \frac{\pi^2}{100}$$

$$\boxed{k \approx \frac{\pi^2}{100} + \frac{(\ln 0.4)^2}{(20)^2} \frac{1}{4}}$$

10) $A = \left(\frac{x}{2}\right)^2 \frac{\pi}{2} + xy$




1

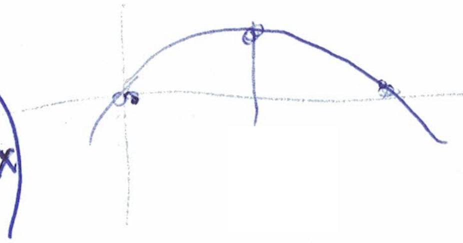
1 $P = 2y + x + \frac{x}{2}\pi = 10$

$\Rightarrow y = 5 - x\left(\frac{\pi}{4} + \frac{1}{2}\right)$

$\Rightarrow A = x^2 \underbrace{\left(\frac{\pi}{8} - \frac{\pi}{4} - \frac{1}{2}\right)}_{-\frac{1}{2} - \frac{\pi}{8}} + 5x$

2

$= x\left(5 - \left(\frac{1}{2} + \frac{\pi}{8}\right)x\right)$



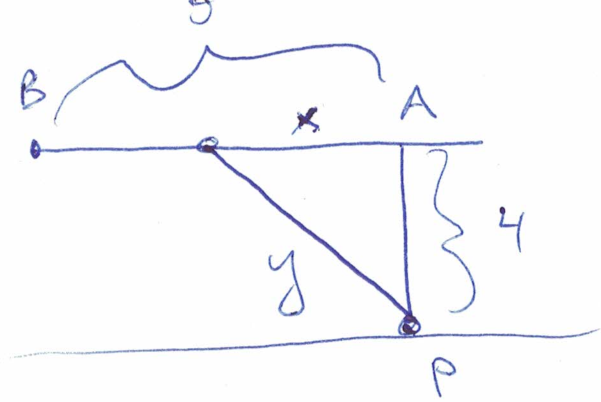
· Roots at $x = 0$, $\frac{5}{\frac{1}{2} + \frac{\pi}{8}}$

· Max at

$x = \frac{1}{2}\left(\frac{5}{\frac{1}{2} + \frac{\pi}{8}}\right)$

$\boxed{x = \frac{5}{1 + \frac{\pi}{4}}}$

11)




Let $x$ & $y$ as shown. Then time it takes to get from P to B [if landing at point $x$] is:

$$T = \frac{y}{4} + \frac{9-x}{5}.$$

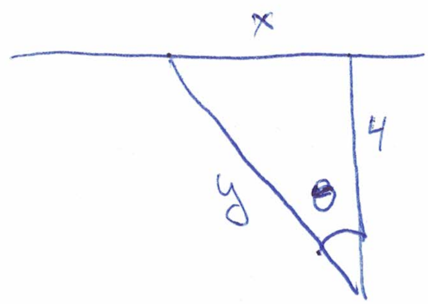
and $y = \sqrt{x^2+4^2}$, $x \in [0,9]$.

$$T' = \frac{x}{4}\frac{1}{\sqrt{x^2+16}} - \frac{1}{5} = 0$$

$$\Rightarrow \quad x^2+16 = \left(\frac{5}{4}x\right)^2 \Rightarrow x^2 = \frac{16}{\frac{25}{16}-1} = \frac{16^2}{9}$$

$$\Rightarrow \boxed{x = \frac{16}{3}}$$

**Or:**




$$y = \frac{4}{\cos\theta}$$

$$x = 4\tan\theta$$

$$T = \frac{y}{4} + \frac{9 - x}{5}$$

$$= \frac{1}{\cos\theta} + \frac{9}{5} - \frac{4}{5}\tan\theta$$

$$T' = \frac{+\sin\theta}{\cos^2\theta} - \frac{4}{5}\sec^2\theta = 0$$

$$\Rightarrow \sin\theta = \frac{4}{5}$$

$$\Rightarrow \cos\theta = \sqrt{1 - \frac{16}{25}} = \sqrt{\frac{9}{25}} = \frac{3}{5}$$

$$\tan\theta = \frac{4}{3}$$

$$\Rightarrow \boxed{x = \frac{16}{3}}$$