Homework 1

1. Evaluate the following limits or explain why it doesnt exsit.

$$\text{(a)} \quad \lim_{x\to 2} \frac{x-2}{x^2-4}$$
$$\text{(b)} \quad \lim_{x\to 4} \frac{\sqrt{x}-2}{x-4}$$
$$\text{(c)} \quad \lim_{x\to 1} \frac{x^3-1}{x-1}$$
$$\text{(d)} \quad \lim_{t\to -1} \frac{t^2}{t+1}$$

Solution.

(a) $\frac{x-2}{x^2-4} = \frac{1}{x+2} \to \frac{1}{4}$ as $x \to 2$.

(b) $\frac{\sqrt{x}-2}{x-4} = \frac{(\sqrt{x}-2)(\sqrt{x}+2)}{(x-4)(\sqrt{x}+2)} = \frac{1}{\sqrt{x}+2} \to \frac{1}{4}$ as $x \to 4$.

(c) $x^3-1 = (x-1)\left(x^2+x+1\right)$ so the limit is 3.

(d) It's $\pm\infty$ (or "does not exist").

2. Find the following limits (they may be infinite).

$$\text{(a)} \quad \lim_{x\to\infty} \frac{x^3+x-5}{3+3x^3}$$
$$\text{(b)} \quad \lim_{x\to-\infty} \frac{x^3+x-5\sin x}{3+3x^3}$$
$$\text{(c)} \quad \lim_{x\to\infty} \frac{x^2}{\sqrt{3+3x^3}}$$
$$\text{(d)} \quad \lim_{x\to\infty} \frac{x^{3/2}}{\sqrt{3+3x^3}}$$
$$\text{(e)} \quad \lim_{x\to\infty} \frac{x}{\sqrt{3+3x^3}}$$
$$\text{(f)} \quad \lim_{x\to\infty} \frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{4x+2}}$$
$$\text{(g)} \quad \lim_{x\to\infty} \sqrt{x+1}-\sqrt{x}$$

Solutions.

(a) $\frac{1}{3}$ (b) $\frac{1}{3}$

(c) $\frac{x^2}{\sqrt{3+3x^3}} \sim \frac{x^2}{\sqrt{3}x^{3/2}} \sim \frac{1}{\sqrt{3}}x^{1/2} \to \infty$ as $x \to \infty$

(d) $\frac{x^{3/2}}{\sqrt{3+3x^3}} \sim \frac{x^{3/2}}{\sqrt{3}x^{3/2}} \sim \frac{1}{\sqrt{3}}$ as $x \to \infty$

(e) 0 (similar to (d))

(f) $\frac{\sqrt{x}+\sqrt{x-3}}{\sqrt{4x+2}} \sim \frac{2x^{1/2}}{2x^{1/2}} \to 1$ as $x \to \infty$

(g) $\sqrt{x+1}-\sqrt{x} = \frac{(\sqrt{x+1}-\sqrt{x})(\sqrt{x+1}+\sqrt{x})}{(\sqrt{x+1}+\sqrt{x})} = \frac{1}{(\sqrt{x+1}+\sqrt{x})} \sim \frac{1}{2}x^{-1/2} \to 0$ as $x \to \infty$

3. (a) Consider

$$\lim_{x\to 0^+} \frac{\left(x^{1/2}+2x^2\right)^{1/2}}{x^p}.$$

Find p such that this limit exists and is non-zero. What is its value?

(b) Repeat part (a) but with "$x \to 0^+$" replaced with "$x \to +\infty$".

Solution. (a) $p = 1/4$, limit is 1. (b) $p = 1$, limit is $\sqrt{2}$.

4. (a) Consider the function

$$f(x) = \frac{1}{1 - x^2}.$$

Find the limits

$$\lim_{x\to 1^+} f(x), \quad \lim_{x\to 1^-} f(x), \quad \lim_{x\to\infty} f(x)$$

Sketch the graph of the function $f(x)$ (note the following useful property: $f(x) = f(-x)$). Indicate vertical and horizontal asymptotes.

(b) Consider the function

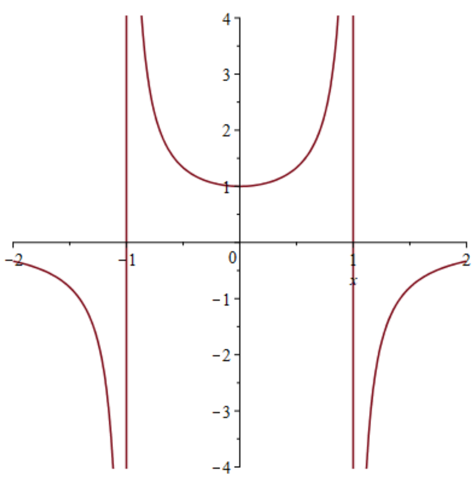
$$f(x) = \frac{1}{(1 - x)^2}.$$

Find the limits

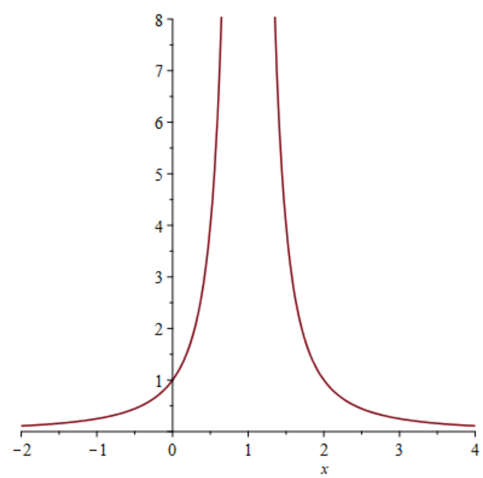
$$\lim_{x\to 1^+} f(x), \quad \lim_{x\to 1^-} f(x), \quad \lim_{x\to\infty} f(x)$$

Sketch the graph of the function $f(x)$. Indicate vertical and horizontal asymptotes.

Solution. Here are the sketches:



(a)



(b)

Looking at the sketches (or otherwise), for part (a) we have

$$\lim_{x\to 1^+} f(x) = -\infty, \quad \lim_{x\to 1^-} f(x) = +\infty, \quad \lim_{x\to\infty} f(x) = 0.$$

whereas for part (b) we have

$$\lim_{x\to 1^+} f(x) = +\infty, \quad \lim_{x\to 1^-} f(x) = +\infty, \quad \lim_{x\to\infty} f(x) = 0.$$

5. Suppose that $\sqrt{5 - 2x^2} \le f(x) \le \sqrt{5 - x^2}$ for $x \in (-1, 1)$. Use the Squeeze theorem to prove that $\lim_{x\to 0} f(x) = \sqrt{5}$.

Solution. The key finding is that both $\sqrt{5 - 2x^2}$ and $\sqrt{5 - x^2}$ converge to the *same* limit ($\sqrt{5}$) as $x \to 0$. So by squeeze theorem, so does $f(x)$.

6. Use the Squeeze theorem to evaluate $\lim_{x\to 0} x\cos(1/x)$.

   Solution. We have $-|x| \le x\cos(1/x) \le |x|$ and both $\pm|x| \to 0$ as $x \to 0$ so then by squeeze theorem $x\cos(1/x) \to 0$ as $x \to 0$.

7. Here are two methods to compute $\sqrt{2} \approx 1.414213562$ numerically. The first is called "Newton iteration". Start with $x_0 = 2$ and define, iteratively,

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}. \tag{1}$$

   The second method is to start with $x_0 = 2$ and define, iteratively,

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}. \tag{2}$$

   (a) Starting with $x_0 = 2$, apply the iteration (1) several times (using calculator/computer). Record your answers to at least 8 decimal digits. How many iterations did you need to get 8 decimal places correctly?

   (b) Starting with $x_0 = 2$, apply the iteration (2) 10 times. How many correct decimal places did you get?

   (c) Based on your observations in parts (a) and (b), estimate how many iterations you will need to get 1000 decimal places of$\sqrt{2}$ using either (1) or (2).

   (d) Solve the equation $x = \frac{x+2}{x+1}$. What does this tell you about (2)?

8. [BONUS] The set of all algebraic numbers $\mathcal{A}$ is defined to be a set of all roots of all polynomials with integer coefficients. That is,

$$\mathcal{A} = \{x : a_n x^n + a_{n-1}x^{n-1} + ... + a_0 = 0, \quad n \in \mathbb{N},\ a_0, a_1, ...a_n \in \mathbb{Z}.\}$$

   (a) Show that $\sqrt{2}, \sqrt{3} + \sqrt{2}, \left(\sqrt{3} + \sqrt{2} + 1\right)^{1/3} \in \mathcal{A}$.

   (b) Show that $\mathcal{A}$ is a countable set.

   Note: a real number that is not algebraic is called transcendental. Since $\mathcal{A}$ is countable but $\mathbb{R}$ is not, there are uncountably many transcendental numbers. Despite their abundance, it is difficult to come up with even one simple example of a transcendental number.

9. [BONUS] Show that there uncountably many letters "L", all of the same size, that can be fit into a plane without intersecting each-other. Show that there is only countably many letters "O" for which this can be done. What about the letter "X"? What if you are allowed to resize it?

Q7 solution

1a) – See Maple worksheet attached.

– Each iteration doubles the # of correct digit

– x_4 is correct to > 8 digits

1b) – Got roughly 8 digits after ~~each~~ 10 iteration

(only 4 digits after 5 iterations)

1c) Since $2^{10} \approx 1000$ and each iteration of (a) doubles # of correct digits, roughly 10 iterations will be enough using (a).

On the other hand, we got 8 digits after 10 iterations of (b) and from the data, we observe that every 5 iterations give additional 4 digits, so we'll need roughly

$$1000 \cdot \frac{5}{4} \approx \boxed{1250} \text{ iterations}$$

of (b) to get 1000 digits

1d) $x = \frac{x+2}{x+1} \iff x^2 = 2 \iff x = \pm\sqrt{2}$.

This implies that if (b) converges $x_n \to x$, then it must converge to either $\sqrt{2}$ or $-\sqrt{2}$.

```
> Digits := 40;
```

$$Digits := 40$$

```
> x := 2.0;
```

$$x := 2.0$$

1.a)

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> to 10 do x := (x^2+2)/2/x; end;
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$x :=$ 1.500000000000000000000000000000000000000 ← x_1
$x :=$ 1.416666666666666666666666666666666666666 ← x_2
$x :=$ 1.414215686274509803921568627450980392157 ← x_3
$x :=$ 1.414213562374689910626295578890134910116
$x :=$ 1.414213562373095048801689623502530243615
$x :=$ 1.414213562373095048801688724209698078570
$x :=$ 1.414213562373095048801688724209698078570 ← Each iteration doubles the # of correct digits
$x :=$ 1.414213562373095048801688724209698078570
$x :=$ 1.414213562373095048801688724209698078570
$x :=$ 1.414213562373095048801688724209698078570

1b)

```
> x := 2.0;
> to 10 do x := (x+2)/(x+1); end;
```

$$x := 2.0$$

$x :=$ 1.333333333333333333333333333333333333333
$x :=$ 1.428571428571428571428571428571428571429
$x :=$ 1.411764705882352941176470588235294117647
$x :=$ 1.414634146341463414634146341463414634146
$x :=$ 1.414141414141414141414141414141414141414 ← x_5
$x :=$ 1.414225941422594142259414225941422594142
$x :=$ 1.414211438474870017331022530329289428076
$x :=$ 1.414213926776740847092605886575735821967
$x :=$ 1.414213499851323223312518584597085935177 ← x_9
$x :=$ 1.414213573100135484665599211725581968223 ← x_{10}

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>
```

~8 digits after 10 iterations

Q8 solutions

3) a) $x = \sqrt{2} \Rightarrow x^2 = 2, \quad x^2 - 2 = 0$

$x = \sqrt{2} + \sqrt{3} \Rightarrow x^2 = 5 + 2\sqrt{6}, \ (x^2 - 5)^2 = 24$

$\Rightarrow \boxed{(x^2 - 5)^2 - 24 = 0}$

Now if $p(x) = 0$ then let ~~$q(y) = p$~~

Let $y = x + 1$, let $q(y) = p(y-1) = p(x)$

then $q(y)$ is a polynomial and has $x+1$ as its root.

Thus $\sqrt{2} + \sqrt{3} + 1$ is algebraic.

Similarly, if $p(x) = 0$ and $y = x^{\frac{1}{3}}$

then let $q(y) = p(y^3) = p(x) = 0$;

$q(y)$ is a polynomial and has $x^{\frac{1}{3}}$ as its root, thus $(\sqrt{2} + \sqrt{3} + 1)^{\frac{1}{3}}$ is algebraic.

3b) Note that $A = A_1 \cup A_2 \cup A_3 \cup \cdots$

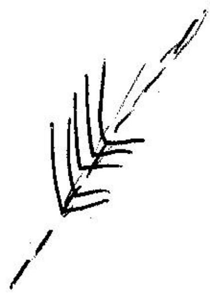
Where $A_n = \{ x :$ a root of polynomial of degree n with integer coefficients $\}$.

Now each A_n is <u>countable</u> and a union of countable sets is also countable

$\Rightarrow A$ is countable.

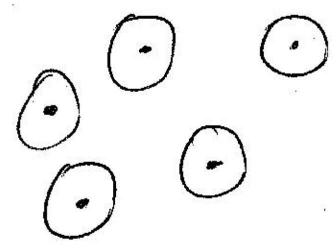
Q9 solutions

4) Stack letters L like this:



For each point on the dashed line, there corresponds a unique letter L. Since ~~there are~~ the set of points along a straight line is uncountable, so is the number of "L" arranged in this way.

For letter "O" the inside of each "O" does not have any other letters; thus we can associate a pair of rational numbers to each "O". Since there are countably many pairs of rationals, there are countably many "O".



For letter X, countably many if they are all of the same size. [Similar argument as "O":]