Homework 4

1. The Super-Douper car can go from zero to 60km/hr in 3sec. (a) Assuming its acceleration is constant, what is it? (b) Assume that the acceleration has the form $a = a_0 t$. Find a_0. What is the acceleration at $t = 3$ in this case?

   Solution. (a) We have $v' = a = const$ so that $v = at$. Moreover $v(3) = 60$ km/hr $= 16.66$ m/sec, so that $3a = 16.666$, or $a = 5.55\text{m/sec}^2$. (b) Here, we have $v' = a_0 t$ so that $v = a_0 \frac{t^2}{2}$, and setting $t = 3$ we get $16.66 = a_0 3^2/2$ so that $a_0 = 3.7037$ and $a = 11.11$ m/sec^2 at $t = 3$.

2. A car travelling 100km/hr hits a break which supplies a constant deceleration of a km/hr^2. (a) How long does it take for a car to stop? (b) What's the "stopping distance", that is, the distance it travels between the time that the breaks were hit and a complete stop?

   Solution. We have $a(t) = -a$ so that $v(t) = -at + 100$. Solving $v(t) = 0$ we obtain $t = 100/a$. (b) Integrating $v(t)$, we get $p(t) = -\frac{at^2}{2} + 100t$. Setting $t = 100/a$ yields $p = \frac{100^2}{2a} = 5000/a$.

3. A rocket of mass $1kg$ is launched vertically upward from the ground with zero initial velocity. The rocket engine supplies a constant upward force of F Newtons for T seconds. After T sec the rocket is subject only to the force of gravity. (a) Find the position and velocity of the rocket when the rocket engine stops. (b) Find the maximum height attained by the rocket. (c) Give the maximum height explicitly for when $T = 10\,\text{sec}$, and $F = 20$ Newtons. (Neglect the air resitance and assume the accelearation due to gravity $g = -9.8m/s^2$ is constant).

   Solution. (a) We have $mv' = F - gm$ subject to initial conditions $v(0) = 0,\ x(0) = 0$, so that, with $m = 1$, we get $v = (F - g)\,t$ and $x = (F - g)\,\frac{t^2}{2}$. The engine stops at $t = T$ at which point, the position and acceleration is given by

$$v(T) = (F-g)\,T, \quad x(T) = (F-g)\,\frac{T^2}{2}. \tag{1}$$

   For $t > T$, we have

$$mv' = -gm \ \text{ subject to initial conditions (1)}$$

   This yields

$$\begin{aligned} v(t) &= (F-g)\,T - g\,(t-T)\,, \quad t \geq T; \\ x(t) &= (F-g)\,\frac{T^2}{2} + (F-g)\,T\,(t-T) - g\frac{(t-T)^2}{2}, \quad t \geq T. \end{aligned}$$

   The maximum height is acheived at time $t_{\max}$ when $v = 0:$

$$\begin{aligned} 0 &= (F-g)\,T - g\,(t_{\max} - T) \\ t_{\max} &= T + \frac{(F-g)\,T}{g} = \frac{FT}{g} \end{aligned}$$

   The height at this time is

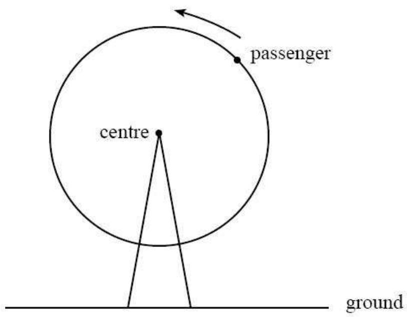
$$\begin{aligned} x(t_{\max}) &= (F-g)\,\frac{T^2}{2} + (F-g)\,T\frac{(F-g)\,T}{g} - g\frac{\left(\frac{(F-g)T}{g}\right)^2}{2} \\ &= (F-g)\,T^2\left(\frac{1}{2} + \frac{F-g}{g} - \frac{(F-g)}{2g}\right) \\ &= \frac{(F-g)\,T^2 F}{2g} \end{aligned}$$

   (c) Plugging in $F = 20, T = 10, g \approx 10$ we get $x_{\max} \approx 1000\text{m=1km}$.

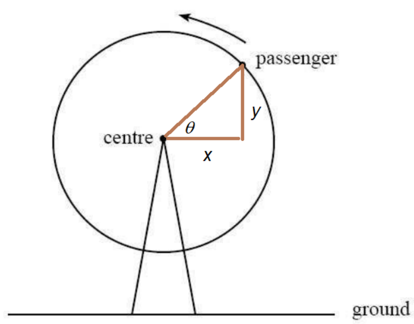
4. A lump of clay is being rolled out so that it maintains the shape of a circular cylinder (and its volume remains constant). If the length is increasing at a rate proportional to itself, show that the radius is decreasing at a rate proportional to itself. Note: $V = \pi r^2 l$.

   Solution. We have $\frac{dV}{dt} = \pi 2r \frac{dr}{dt} l + \pi r^2 \frac{dl}{dt}$. Since the volume remains constant, $\frac{dV}{dt} = 0$, so that $\frac{dr}{dt} = -\frac{r}{2l}\frac{dl}{dt}$. "The length is increasing at a rate proportional to itself" means $\frac{dl}{dt} = Cl$ for some constant C, so that then $\frac{dr}{dt} = -\frac{C}{2}r$, i.e. $\frac{dr}{dt}$ is decreasing at a rate proportional to itself.

5. A circular ferris wheel with radius 10 metres is revolving at the rate of 10 radians per minute. How fast is a passenger on the wheel rising when the passenger is 6 metres higher than the centre of the wheel and is rising?

   Solution. Let x, y, θ be as indicated here:

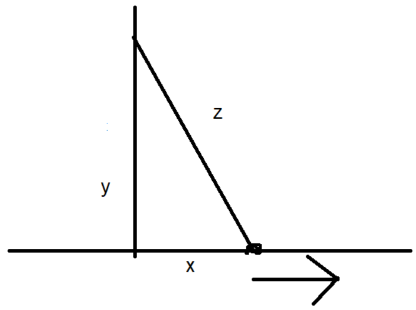
   

   We are given that $\frac{d\theta}{dt} = 10$, and we have $y = 10\sin\theta$. Then $\frac{dy}{dt} = 10\cos\theta\frac{d\theta}{dt}$. Moreover when $y = 6$, we have $x = \sqrt{10^2 - 6^2} = 8$, so that $\cos\theta = 8/10$. Therefore at that point, $\frac{dy}{dt} = 10 \times (8/10) \times 10 = 80$.

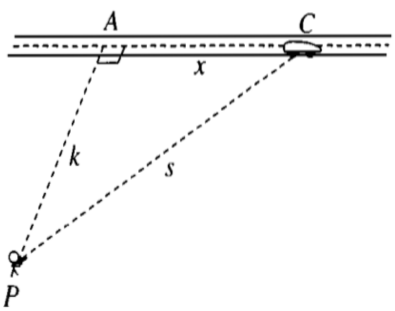
6. The top of a ladder 5m long rests against a vertical wall. If the base of the ladder is being pulled away from the base at a rate of 1/2 m/s, how fast is the top of the ladder slipping down the wall when it is 3m above the base of the wall?

   Solution. See diagram below. We have $x^2 + y^2 = 5^2$ so that $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. Given that $\frac{dx}{dt} = 5$

when $y = 3$. At that time, $x = 4$ so that $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{4}{3} \times \frac{1}{2} = -2/3$.

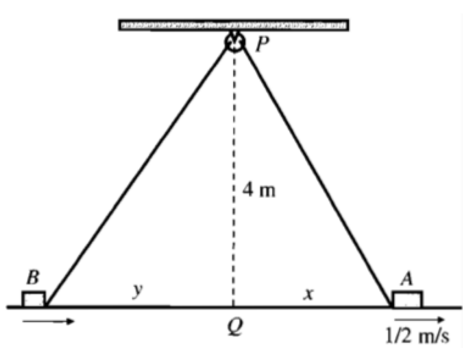


7. A policeman is standing near a highway using a radar gun to catch speeders. He aims the gun at a car that has just passed his position and, when the gun is pointing at an angle of 45^o to the direction of the highway, notes that the distance between the car and the gun is increasing at a rate of 100km/h. (a) How fast is the car travelling? (b) If the radar gun is aimed at a car travelling 90km/h along a straight road, what will the reading be when it is aimed making an angle of 30^o with the road?

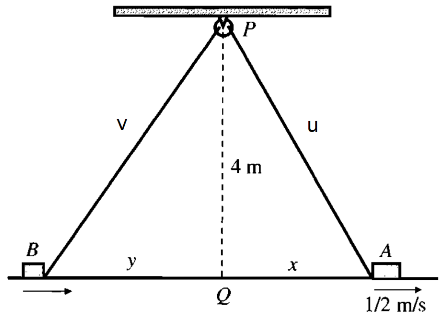


Solution. (a) We have $s' = 100$ and $x(t)^2 + k^2 = s(t)^2$. Differentiating gives us $2xx' = 2ss'$. Moreover $s = \sqrt{2}x$ (because 45 degrees!) which yields $x' = \sqrt{2}100 = 141$km/hr. (b) Here, we also have $xx' = ss'$ but now, $s = \frac{2}{\sqrt{3}}x$ and $x' = 90$ yields $s' = 90\frac{\sqrt{3}}{2} \approx 78$ km/hr.

8. Two crates A and B are on the floor of a warehouse and are joined by a tight rope 15m long through the pulley in the ceiling of height 4m as shown in figure below. When the crate A is 3m from the point below the pully and is being moved at speed $\frac{1}{2}m/s$, how fast is crate B is moving?



Solution. Let u, v be as shown in the figure:



Then we have the following relationships:

$$x^2 + 4^2 = u^2;$$
$$y^2 + 4^2 = v^2;$$
$$u + v = 15.$$

Differentation yields:

$$xx' = uu'; \quad yy' = vv';$$
$$u' = -v'.$$

Moreover at the time when $x = 3$, $x' = 1/2$, we solve these eqations to get $u = 5$, $v = 10, y = \sqrt{84} = 9.16$, so that

$$3/2 = 5u', \quad \sqrt{85}y' = 10v', \ u' = -v'$$

and so finally

$$y' = -3/\sqrt{84} = -0.327.$$