Homework 5

1. Population tends to grow at a rate roughly proportional to the population present. The population of the US was approximately 179 million in 1960 and 205 million in 1970. (a) Use this information to estimate the population in 1940. (b) According to this model, when would US population exceed 300 million? (c) Use google to look up the actual US population. Comment on how well did the model do.

2. A pesticide sprayed onto tomatoes decomposes into a harmless substance at a rate proportional to the amount of the pesticide $M(t)$ still unchanged at time t. If the initial amount of 10 points is sprayed onto an acre reduces to 5 pounds in 6 days, when will 80% of the pesticide be decomposed?

3. The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and the half-life of radium (the time required for half of the substance to decay) is 1690 years, how much radium will be present 100 years from now?

4. During a cold night, the heating in the house broke down. The house cooled from 20 degrees to 15 degrees in 2 hours. The outside temperature is -20 degrees. How long will it be before the temperature in the house goes down to 10 degrees?

5. A marble rolls along a straight line in such a manner that its velocity is directly proportional to the distance it has *yet* to roll. If the total distance it rolls is 5 meters, and after 1 second it has rolled 2 meters, express the distance it has rolled as a function of time. How long will it take for the marble to roll 4 meters?

6. A certain fish reproduces at a rate proportional to its total population $y(t)$. Moreover, it is harvested at a rate of of 2 million fish per year. Thus, $y(t)$ (measured in millions) satisfies

$$\frac{dy}{dt} = ry - 2$$

   where r is its reproduction rate. When harvesting began, there were estimated to be 6 million fish in a lake. After one year, only 5 million fish remained. (a) How many fish will remained after two years? (a) If the harvesting continues unchecked, how long will it be before no fish remain? (b) How should you adjust the harvesting rate to prevent extinction of the fish?

7. Find derivatives of the following functions: (a) $y = x^x$ (b) $y = x^{1/x}$ (c) $y = x^{x^x}$.

8. In class, we started with $\ln x$ and then defined e^x as its inverse. In this exercise, we will define e^x first, then define $\ln x$ as its inverse. To define e^x, assume that there exists a *unique* differentiable function $f(x)$ that is defined for all x and that satisfies the equation,

$$f'(x) = f(x) \text{ and } y(0) = 1. \tag{1}$$

   (a) Show that $f(x + y) = f(x)f(y)$. Hint: use the fact that the solution to (1) is unique.
   (b) Show that $f(-x) = 1/f(x)$.
   (c) [BONUS]: show that $f(x) > 0$ for all x.
   (d) Show that $f(x)$ is increasing. Hint: use part (c).
   (e) Let $g(x)$ be the inverse of $f(x)$. [such an inverse exists since $f(x)$ is increasing]. Show that $g'(x) = \frac{1}{x}$ and that $g(1) = 0$.
   We then call $f(x)$ the "exponential", $f(x) = e^x$, and $g(x)$ the "logarithm", $g(x) = \ln(x)$.
   (f) Show that $f(x) \geq 1$ for $x \geq 0$, then show that $f(x) \geq 1 + x$ for $x \geq 0$
   (g) We define $e = f(1)$. Using part (a), show that $f(x) = e^x$, at least for integer x.
   (h) Show that $e^x \to \infty$ as $x \to \infty$ and that $f(x) \to 0$ as $x \to -\infty$.

9. The function $y = y(x)$ is defined implicitly by the equation $\sin x + \cos(\ln(y)) + y^2 x = 1$. Determine $\frac{dy}{dx}$ at the point $x = 0, y = 1$.

10. You are given a function $f(x)$ that satisfies

$$\frac{d}{dx}[f(x)] = \frac{1}{x^3+1}, \quad f(1) = 2.$$

Let g be the inverse of f; that is, $f(g(x)) = x$. Determine $g(2)$ and $g'(2)$.

11. The hyperbolic trigonomentric functions are defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}; \quad \tanh x = \frac{\sinh x}{\cosh x}$$

(a) Verify the following identities:

$$\frac{d}{dx}\cosh x = \sinh x; \quad \frac{d}{dx}\sinh x = \cosh x;$$
$$\cosh^2 x - \sinh^2 x = 1.$$
$$\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$$

(b) Derive an addition formula for $\cosh(x+y)$ which is similar to the addition formula for $\cos$.

(c) Sketch the graphs of $\sinh x, \cosh x$ and $\tanh x$. Indicate any odd or even symmetry.