**Homework 5**

1. Population tends to grow at a rate roughly proportional to the population present. The population of the US was approximately 179 million in 1960 and 205 million in 1970. (a) Use this information to estimate the population in 1940. (b) According to this model, when would US population exceed 300 million? (c) Use google to look up the actual US population. Comment on how well did the model do.

   **Solution.** This is exponential growth, $y = Ce^{rt}$ where $y$ is population size, $r$ is growth rate and $C$ is some constant. Let's take 1960 to be $t = 0$. Then $y(0) = 179$ and $y(10) = 205$, so that $C = 179$ and $e^{10r} = 205/179$ so that $r = 0.01356$. (a) 1940 corresponds to $t = -20$ at which point

   $$y(-20) = 179e^{-0.271} = 136.5$$

   So population in 1940 is about 136.5 million. (b) Solving $y = 300$ we obtain $t = \frac{\ln(300/179)}{r} = 38.07$ which corresponds to the year 1998. (c) The year 2018 corresponds to $t = 58$. Then $y(58) = 393$. Actually, according to google, the US population in 2018 is 323 million. So the exponential model over-estimates the population size significantly this far out (by 21%). (Another point of comparison is 1998, when the model predicts US population to exceed 300 million, whereas the actual population that year was 282 million; overestimation by only 6.4%).

2. A pesticide sprayed onto tomatoes decomposes into a harmless substance at a rate proportional to the amount of the pesticide $M(t)$ still unchanged at time $t$. If the initial amount of 10 points is sprayed onto an acre reduces to 5 pounds in 6 days, when will 80% of the pesticide be decomposed?

   **Solution.** Here, we have exponential decay $M = Ce^{-rt}$ with $C = 10$, $10e^{-6r} = 5$ so that $6r = \log(2) \implies r = 0.1155$. We want $t$ such that $e^{-rt} = 0.2$ which gives $t = -\log(0.2)/r = 13.9$. So it takes 13.9 days.

3. The rate of decay of radium is proportional to the amount present at any time. If 60 mg of radium are present now and the half-life of radium (the time required for half of the substance to decay) is 1690 years, how much radium will be present 100 years from now?

   **Solution.** We have $y = 60e^{-rt}$ and half-life of 1690 years meants that $e^{-r1690} = 0.5$, Solution is $r = 0.0004101$, and in 100 years there will be $60e^{-r100} = 57.59$ mg of radium left.

4. During a cold night, the heating in the house broke down. The house cooled from 20 degrees to 15 degrees in 2 hours. The outside temperature is -20 degrees. How long will it be before the temperature in the house goes down to 10 degrees?

   **Solution.** Newton's law of cooling says that $y' = -ry$, where $y = T - (-20)$ is the difference between the outside and house temp. Moreover $y(0) = 40$ and $y(2) = 35$, so that $y = 40e^{-rt}$ with $40e^{-2r} = 35$, so that $r = 0.0676$. Temp will be 10 degrees inside when $y = 30$. So we solve $30 = 40e^{-0.0676t}$ for $t$ to obtain $t = 4.25$ hours.

5. A marble rolls along a straight line in such a manner that its velocity is directly proportional to the distance it has *yet* to roll. If the total distance it rolls is 5 meters, and after 1 second it has rolled 2 meters, express the distance it has rolled as a function of time. How long will it take for the marble to roll 4 meters?

   **Solution.** We have $y' = -ay$, where $y$ is the distance yet to roll. The minus sign is because it's rolling towards the origin. And we have $y(0) = 5$, $y(1) = 3$, so that $y = 5e^{t\log(3/5)}$. It will roll 4 meters when $y = 1$. So we solve for $t$ :

   $$1 = 5e^{t\log(3/5)} \implies t = \frac{\log(1/5)}{\log(3/5)} = 3.15.$$

6. A certain fish reproduces at a rate proportional to its total population $y(t)$. Moreover, it is harvested at a rate of of 2 million fish per year. Thus, $y(t)$ (measured in millions) satisfies

   $$\frac{dy}{dt} = ry - 2$$

where $r$ is its reproduction rate. When harvesting began, there were estimated to be 6 million fish in a lake. After one year, only 5 million fish remained. (a) How many fish will remained after two years? (b) If the harvesting continues unchecked, how long will it be before no fish remain? (c) How should you adjust the harvesting rate to prevent extinction of the fish?

**Solution.** (a) We need to solve the ODE. Write $ry - 2 = r(y - 2/r)$ and change variables, $u = y - 2/r$. Then $du/dt = ru$ so that $u = Ce^{rt}$ and

$$y = Ce^{rt} + 2/r.$$

Plug in $y(0) = 6$, $y(1) = 5$:

$$\begin{aligned} 6 &= C + 2/r \\ 5 &= Ce^r + 2/r \end{aligned}$$

so that

$$C = 6 - 2/r$$

and

$$5 = (6 - 2/r)\, e^r + 2/r. \tag{1}$$

The equation (1) cannot be solved explicitly and must be solved numerically. We obtain

$$r = 0.181, \quad C = -5.03$$

Then $y(2) = Ce^{2r} + 2/r = 3.801$. So 3.8 million fish remain after two years.

(b) All fish will be gone when $y = 0$ or $t = \frac{1}{r}\log(\frac{-2}{Cr}) = 4.33$ years.

(c) For a general harvesting rate, we have

$$\frac{dy}{dt} = ry - h.$$

The harvesting and growth are precisely in a balance when $ry - h = 0$. But then $y = h/r$ is a constant so that $y(0) = y(t) = h/r$. So we choose $h = ry(0) = 1.086$. When $h > 1.086$, all fish go extinct eventually. When $h < 1.086$, fish population grows. So we must have $h \le 1.086$ to insure fish survival.

7. Find derivatives of the following functions: (a) $y = x^x$ (b) $y = x^{1/x}$ (c) $y = x^{x^x}$.

(a) $y = \exp(x \log x)$ so that

$$\begin{aligned} y' &= (x \log x)' \exp(x \log x) \\ &= (\log x + 1) \exp(x \log x). \end{aligned}$$

(b) $y = \exp\left(\frac{1}{x}\log x\right)$ so that

$$\begin{aligned} y' &= \left(\frac{1}{x}\log x\right)' \exp\left(\frac{1}{x}\log x\right) \\ &= \left(\frac{1}{x^2} - \frac{1}{x^2}\log x\right)' \exp\left(\frac{1}{x}\log x\right) \end{aligned}$$

(c) $y = \exp\left(\log x^{x^x}\right) = \exp(x^x \log x)$ so

$$\begin{aligned} y' &= (x^x \log x)' \exp(x^x \log x) \\ &= \left((x^x)' \log x + \frac{x^x}{x}\right) \exp(x^x \log x) \\ &= \left((\log x + 1)\exp(x \log x) \log x + x^{x-1}\right) \exp(x^x \log x) \\ &= \left((\log x + 1)\log x + x^{-1}\right) x^x x^{x^x} \end{aligned}$$

8. In class, we started with $\ln x$ and then defined $e^x$ as its inverse. In this exercise, we will define $e^x$ first, then define $\ln x$ as its inverse. To define $e^x$, assume that there exists a *unique* differentiable function $f(x)$ that is defined for all $x$ and that satisfies the equation,

$$f'(x) = f(x) \text{ and } y(0) = 1. \tag{2}$$

(a) Show that $f(x+y) = f(x)f(y)$. Hint: use the fact that the solution to (2) is unique.

(b) Show that $f(-x) = 1/f(x)$.

(c) [BONUS]: show that $f(x) > 0$ for all $x$.

(d) Show that $f(x)$ is increasing. Hint: use part (c).

(e) Let $g(x)$ be the inverse of $f(x)$. [such an inverse exists since $f(x)$ is increasing]. Show that $g'(x) = \frac{1}{x}$ and that $g(1) = 0$.
We then call $f(x)$ the "exponential", $f(x) = e^x$, and $g(x)$ the "logarithm", $g(x) = \ln(x)$.

(f) We define $e = f(1)$. Using part (a), show that $f(x) = e^x$, at least for integer $x$ (in fact this is true for all real $x$).

(g) Show that $f(x) \geq 1$ for $x \geq 0$, then show that $f(x) \geq 1 + x$ for $x \geq 0$. Conclude that $e \geq 2$.

(h) Show that $e^x \to \infty$ as $x \to \infty$ and that $f(x) \to 0$ as $x \to -\infty$.

**Solutions.** (a) Let $g(x) = \frac{f(x+y)}{f(y)}$. Then $g'(x) = \frac{f'(x+y)}{f(y)} = \frac{f(x+y)}{f(y)} = g(x)$, and $g(0) = 1$. So both $g$ and $f$ satisfy (2), and therefore by uniqueness, $g = f$.

(b) By part (a), $f(x)f(-x) = f(x-x) = f(0) = 1$.

(c) Suppose that $f(x)$ is negative somewhere. Then let $x$ be the first positive root of $f$ (it exists by intermediate value theorem). But then by mean value theorem, there exists $c \in (0, x)$ such that $f'(c) = \frac{f(x)-f(0)}{x} = -1/x < 0$. But then $f(c) = f'(c) < 0$, which contradics the fact that $x > c$ is the first positive root of $f$.

(d) Since $f$ is positive, then $f' = f$ is also positive everywhere, so $f$ is increasing.

(e) The inverse satisfies: $f(g(x)) = x$. Differentiating we get

$$\begin{aligned} f'(g(x))g'(x) &= 1, \\ f(g(x))g'(x) &= 1, \\ xg'(x) &= 1 \\ g'(x) &= \frac{1}{x}. \end{aligned}$$

(f) This is a direct consequence of (a). For example $f(2) = f(1+1) = f(1)^2 = e^2$, $f(3) = f(2+1) = e^2 f(1) = e^3$ etc.

(g) $f(x)$ is increasing and $f(0) = 1$, so then $f(x) \geq 1$ for $x \geq 0$. Next, $f'(x) = f(x) \geq 1$, which means that $f(x) \geq x + f(0) = x + 1$. Then $e = f(2) \geq 1 + 1 = 2$.

(h) We showed in (g) that $f(x) \geq 1 + x$, and so $e^x \geq 1 + x \to \infty$ as $x \to \infty$. On the other hand, $e^{-x} = \frac{1}{e^x} \to 0$ as $x \to \infty$.

9. The function $y = y(x)$ is defined implicitly by the equation $\sin x + \cos(\ln(y)) + y^2 x = 1$. Determine $\frac{dy}{dx}$ at the point $x = 0, y = 1$.

**Solution.** Differentiate to get

$$\cos x + (-\sin \ln y)\frac{y'}{y} + 2yy'x + y^2 = 0.$$

Plug in $x = 0, y = 1$ to get

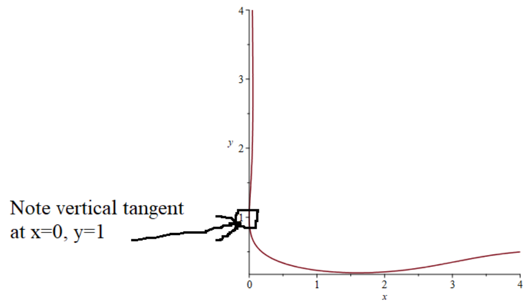
$$1 = 0.$$

Actually *the question has a "bug"*... basically to avoid $1 = 0$ means that $y'$ must be infinite (i.e. vertical slope) at $x = 0$. Here is Maple implicit plot of this relationship that shows that this is indeed the case:

```
> oo:=sin(x)+cos(log(y))+y^2*x-1;
```

$$oo := \sin(x) + \cos(\ln(y)) + y^2 x - 1$$

```
> with(plots):
> implicitplot(oo, x=-4..4, y=-4..4,numpoints=10000);
```




10. You are given a function $f(x)$ that satisfies

$$\frac{d}{dx}f(x) = \frac{1}{x^3+1}, \quad f(1) = 2.$$

Let $g$ be the inverse of $f$; that is, $f(g(x)) = x$. Determine $g(2)$ and $g'(2)$.

**Solution.** $g(2) = 1$; and differentiating we obtain

$$\frac{g'(x)}{g^3(x)+1} = 1$$

so that $g'(2) = g^3(2) + 1 = 2$.

11. The hyperbolic trigonomentric functions are defined as:

$$\cosh x = \frac{e^x + e^{-x}}{2}; \quad \sinh x = \frac{e^x - e^{-x}}{2}; \quad \tanh x = \frac{\sinh x}{\cosh x}$$

(a) Verify the following identities:

$$\frac{d}{dx}\cosh x = \sinh x; \quad \frac{d}{dx}\sinh x = \cosh x;$$
$$\cosh^2 x - \sinh^2 x = 1.$$
$$\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x$$

(b) Derive an addition formula for $\cosh(x + y)$ which is similar to the addition formula for $\cos$.

(c) Sketch the graphs of $\sinh x, \cosh x$ and $\tanh x$. Indicate any odd or even symmetry.

**Solution.** Here, we show addition identities. They are easy to show directly from the definition. Alternatively, note that $f(x) = \sinh(x+y)$ solves the ODE $f'' = f$ subject to initial conditions $f(0) = \sinh(y)$ and $f'(0) = \cosh(y)$. More generally, the solution to $f'' = f$ can be written as $f(x) = A\cosh(x) + B\sinh(x)$, where $A = f(0)$, $B = f'(0)$. Taking $f(0) = \sinh(y)$ and $f'(0) = \cosh(y)$ (and using the fact the solution corresponding to given initial conditions is unique) yields the identity

$$\sinh(x+y) = \sinh(y)\cosh(x) + \cosh(y)\sinh(x).$$

Similar argument shows that

$$\cosh(x+y) = \cosh(y)\cosh(x) + \sinh(y)\sinh(x).$$