MATH 1500, Homework 6

1. Find real numbers x, y such that $z = x + iy$ where (a) $z = \frac{2}{3-4i}$. (b) $z = e^{3+i\pi/4}$.

   Solution. (a) $\frac{2}{3-4i} = \frac{2(3+4i)}{(3-4i)(3+4i)} = \frac{2(3+4i)}{3^2+4^2} = \frac{6}{25} + \frac{8}{25}i$

   (b) $e^{3+i\pi/4} = e^3 e^{i\pi/4} == e^3[\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})] = e^3\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$.

2. (a) Let $z = 1 + \sqrt{3}i$. Find r and θ such that $z = re^{i\theta}$. (b) What is $\left(1+\sqrt{3}i\right)^{2015}$?

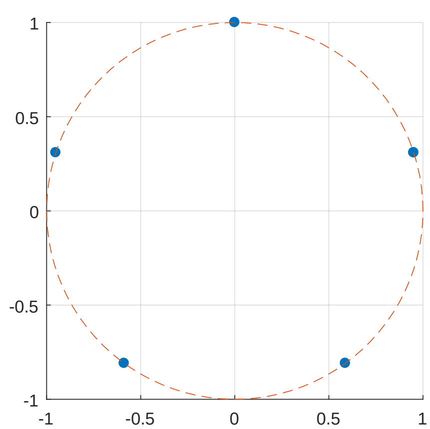
   Solution. (a) The point $\left(1, \sqrt{3}\right)$ is at distance 2 from the origin. The line through $\left(1, \sqrt{3}\right)$ and the origin is at an angle of 60 degrees with the x-axis. Thus $r = 2$ and $\theta = \pi/3$. (b) $\left(1+\sqrt{3}i\right)^{2015} = \left(2e^{i\pi/3}\right)^{2015} = 2^{2015}e^{i\pi 2015/3}$. Now $2015/3 = 672 - 1/3$ so that $e^{i\pi 2015/3} = e^{672\pi i}e^{-i\pi/3} = e^{-i\pi/3}$. Thus $\left(1+\sqrt{3}i\right)^{2015} = 2^{2015}e^{-i\pi/3} = 2^{2015}\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$.

3. Determine all the roots of the polynomial $z^5 - i = 0$, and sketch them in a complex plane.

   Solution. Let $z = re^{i\theta}$. Then $z^5 = r^5 e^{5i\theta} = i = e^{\frac{\pi}{2}i}$. Thus we get $r^5 = 1$ and $5\theta = \frac{\pi}{2} + 2\pi n$, $n = 0 \ldots 4$, or

   $$z = e^{\frac{2\pi n}{5}i}e^{\frac{\pi}{10}i}.$$

   It's a pentagon, rotated by $\pi/10$ degrees off the positive x-axis. Here is the sketch:



4. (a) Find the general solution to the ODE $y'' - y' - 2y = 0$. (b) Find the solution to this ODE subject to initial conditions $y(0) = 0$, $y'(0) = -1$.

   Solution. Try the anzatz $y = \exp(\lambda t)$ to obtain $\lambda^2 - \lambda - 2\lambda = (\lambda+1)(\lambda-2) = 0$ so that the general solution is $y = Ae^{-t} + Be^{2t}$. Initial conditions then yield $A + B = 0$, $\ -A + 2B = -1$ so that $y = \frac{1}{3}\left(e^{-t} - e^{2t}\right)$.

5. (a) Find the general solution to the ODE $4y'' - 4y' + y = 0$. (b) Find the solution to this ODE that in addition satisfies initial conditions $y(0) = 1$, $y'(0) = 2$.

   Solution. Try the anzatz $y = \exp(\lambda t)$ to obtain $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0$. There is a double root $\lambda = 1/2$ so that the general solution is $y = Ae^{t/2} + Bte^{t/2}$. (b) Initial conditions yield $A = 1, A/2 + B = 2$ so that $A = 1, B = 3/2$ and $y = e^{t/2} + \frac{3}{2}te^{t/2}$.

6. (a) Find the general solution to the ODE $y'' - 4y' + 5y = 0$. (b) Find the solution to this ODE that in addition satisfies initial conditions $y(0) = 1$, $y'(0) = 2$.

Solution. Try the anzatz $y = \exp(\lambda t)$ to obtain $\lambda^2 - 4\lambda + 5 = 0 \implies \lambda = 2 \pm i$. So $y = e^{2t}e^{it}$ is a particular solution; and so is its real and imaginary part. Hence $e^{2t}\cos t$ and $e^{2t}\sin t$ are also solutions so that the general solution is $y = Ae^{2t}\cos t + Be^{2t}\sin t$. (b) Initial conditions yield $A + B = 1, 2A + B = 2$ so that $A = 1, B = 0$ and $y = e^{2t}\cos(t)$

7. (a) Write down a second order linear ODE that has the following particular solutions: $y_1 = e^t,\ \ y_2 = e^{-2t}$. (b) Write down a second order linear ODE that has the following particular solution: $y_1 = \cos(t)e^t$. What is another independent particular solution?

   Solution. (a) Roots of characteristic are $\lambda = 1, \lambda = -2$ so the characteristic polynomial is $(\lambda-1)(\lambda+2) = \lambda^2 + \lambda - 2$, corresponding to the ODE $y'' + y' - 2y = 0$. (b) Here, $\lambda = 1 \pm i$, the corresponding characteristic poly is

   $$(\lambda - (1+i))(\lambda - (1-i)) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2$$

   so the ODE is

   $$y'' - 2y' + 2y = 0.$$

8. (a) Find the general solution to the ODE $y''' - y' = 0$. (b) Find the solution to this ODE subject to initial conditions $y(0) = 0,\ y'(0) = 0,\ \ y''(0) = 1$.

   Solution. (a) Here, the characteristic polynomial is $\lambda^3 - \lambda = 0$ and has roots $\lambda = 0, \pm 1$. So general solution is

   $$y = A + Be^t + Ce^{-t}.$$

   (b) The initial conditions then yield the system

   $$\begin{aligned} A + B + C &= 0 \\ B - C &= 0 \\ B + C &= 1 \end{aligned}$$

   which has solution $A = -1, B = C = \frac{1}{2}$ so the solution is

   $$\begin{aligned} y &= -1 + \frac{1}{2}\left(e^t + e^{-t}\right) \\ &= -1 + \cosh t. \end{aligned}$$

9. A mass of $m = 0.1$ kg is attached to a spring. The spring constant is known to be $k = 50N/m$, and you wish to find the friction constant c; recall that the mass motion satisfies $mx'' + cx' + kx = 0$. To find the spring constant, you observe the spring oscillates with a frequency of 3 Hertz. Determine c.

   Solution. Recall that characterisitic equation $m\lambda^2 + c\lambda + k = 0$ has roots $\lambda = -\frac{c}{2m} \pm i\frac{1}{2m}\sqrt{4mk - c^2}$. The solution is then given Kby $x = Ae^{-\frac{c}{2m}t}cos(\omega t - C)$ where A, C are some constants that depend on initial conditions, and where $\omega = \frac{1}{2m}\sqrt{4mk - c^2}$. Frequency of 3 Hertz means three full oscillations per second, so that $\omega = 3 \times 2\pi = 6\pi$. Solving for c we then obtain $c = \sqrt{4mk - 4m^2\omega^2} \approx 2.4058$.