Math 1500, final exam 1 review

How to prepare for the final:

- The most effective way is to go over the problems, especially on homeworks, review sheets and midterms.
- Please attempt the problems *before* looking up the answers. Remember: you learn by doing; math is not a spectator sport!

The semester in a nutshell:

- Limits, continuity, delta-epsilon proofs
- Maximum value theorem and intermediate value theorem
- Definition of derivative, increasing/decreasing functions, mean value theorem
- trigs and their inverses (eg. find derivative of arcsin)
- exp-log [including differentiating things like $\sin x^{\ln x}$]
- Applications of derivatives:
  - Sketching of functions (including things like $x^x = e^{x \ln x}$)
  - Related rates: this includes problems involving trig functions, similar triangles etc
  - Optimization problems
  - Exponential growth / Newton's law of cooling
- linear, quadratic approximations including error estimates
- Taylor series
- areas, Fundamental theorem of calculus, Riemann sums

Some additional review questions:

1. Continuity and limits:
   (a) State delta-epsilon definition of continuity. Show that x^2 is continuous.
   (b) Show that a step function $f(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases}$ is discontinuious at 0.
   (c) Show that the function $f(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is discontinuous at zero.
2. Differentiation from first principles: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$. Example: Find the derivative of $f(x) = \sqrt{1+2x}$ using the definition of derivative. No credit will be given if you use differentiation rules but you can use them to check your answer.
3. See limit questions on review sheets and homeworks. In addition, do the following limits:

$$\lim_{x\to\infty} \frac{\ln x}{x}, \quad \lim_{x\to\infty} e^{-x+3}, \quad \lim_{x\to 0} x \ln x, \quad \lim_{x\to\infty} \frac{e^x}{\ln x}, \quad \lim_{x\to 0} \frac{\sin(2x)}{x}, \quad \lim_{t\to 0} \frac{1-\cos(t)}{\ln(1+t^2)}, \quad \lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)+4x}$$

4. Basic differentiation: for example $\frac{d}{dx}\tan(x\cos 2x)$, inverse trigs (example: $\frac{d}{dx}\arctan(x)$), log differentiation (example: $\frac{d}{dx}\left((\cos x)^{2x}\right)$).

5. Show that the function $f(x) = 2x + \cos(x)$ is one-to-one. Let g be the inverse of f. Find $g(1)$ and $g'(1)$.

6. Suppose that $\tan y + \arctan x = y^2 x + 1$. Find $\frac{dy}{dx}$ at the point $x = 0,\ y = \frac{\pi}{4}$.

7. IVT, MVT etc:

   (a) Using the Intermediate Value Theorem, show that there is at least one positive solution to $x^3 - 1 = x$. Then show that that there is exactly one such solution with $x > 0$ using either Rolle's or Mean Value Theorem.

   (b) Estimate $\ln(3)$ (Hint: start with $\frac{\ln(3)-\ln(e)}{3-e}$ and apply mean value theorem)

   (c) Suppose that $f(0) = 0,\ f(1) = 0$, and $f(2) = 1$. Show that $f'(a) = \frac{1}{2}$ for some $a \in (0,2)$. Show that $f''(b) > \frac{1}{2}$ for some $b \in (0,2)$. Shwo that $f'(c) = \frac{1}{7}$ for some $c \in (0,2)$.

8. Related rates:

   (a) A man 2 m tall walks toward a lamppost on level ground at a rate of 0.5 m/s. If the lamp is 5 m high on the post, how fast is the length of the man's shadow decreasing when he is 3 m from the post? How fast is the shadow of his head moving at that time?

   (b) A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1m higher than the bow of the boat. If the rope is pulled in at a rate of 1 m/s, how fast is the boat approaching the dock when it is 8 m from the dock?

   (c) If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10cm.

9. Graphing: Sketch the graph of a given function $f(x)$. Indicate behaviour as $x \to \pm\infty$, any asymptotes, roots, critical points, max/min. For example, $f(x) = \frac{e^x}{x^2}$; $f(x) = \frac{x}{x^2-4}$; $f(x) = x^{-x}$. See also graphing questions on review1&2, midterm1&2 and homework questions.

10. Max/min:

    (a) A farmer has 300 meters of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fance along the river. What are the dimensions of the field that has the largest area?

    (b) At noon, a sailboat is 20 km due south of a freighter. The sailboat is travelling due east at 20 km/hr, and the freighter is travelling due south at 40 km/hr. When are the two ships closest to each other, and what is the distance between them at that point?

    (c) See max/min questions on review2, midterm2 and homeworks.

11.

    (a) A radioactive substance decays at a rate r and is being replenished at a constant rate h. That is,
    $$\frac{dy}{dt} = -ry + h$$
    Solve this differential equation with initial condition $y(0) = y_0$. Find the limit $\lim_{t\to\infty} y(t)$.

    (b) See midterm 2 questions.

    (a) A particle is moving on a straight line and its acceleration at time t is given by $a(t) = \sqrt{t}$. At time $t = 0$ its position is given by $p(0) = 2$ and its velocity is $v(0) = 3$. Find the the position of the particle at time $t = 1$.

(b) A car is travelling at 25 m/sec when the driver sees an accident 80m ahead and slams on the breaks. What constant acceleration is required to avoid pileup?

12. Linear approximations etc; see also questions on review sheets and homeworks

(a) Use linear approximation to estimate $(25)^{1/3}$. Then find an interval $[a, b]$ to guarantee that $(25)^{1/3} \in [a, b]$.

(b) Use quadratic approximation to estimate $(25)^{1/3}$. Estimate the maximum error.

(c) Estimate $\cos(1)$ using Taylor series around $x = 0$. Use the first three non-zero terms. What is the error in your estimate? Compare with the answer you get from a calculator.

13. See Taylor series questions on homeworks/review sheet.

(a) Write a Taylor series expansion of $f(x) = \frac{\ln(1-x)}{x}$.

(b) Using your answer in (a), show that

$$\int_0^1 \frac{\ln(1-x)}{x} = -1 - \frac{1}{4} - \frac{1}{9} \cdots$$

14. (a) Set up the Riemann sum to approximate the area under the curve $y = \frac{1}{x^3+1}$ between $x = 1$ and $x = 2$ with $n = 3$ subintervals and with the sample points taken at the left endpoints. Sketch a diagram to illustrate.

(b) Show that $\int_1^2 \dfrac{1}{x^3+1} dx \leq 3/8$. (hint: first show that $\dfrac{1}{x^3+1} \leq \dfrac{1}{x^3}$)

(c) Suppose that $g(x) = \displaystyle\int_{\ln(x)}^{x^2-1} \frac{1}{\sqrt{t+t^2}} dt$. Find $g(1)$ and $g'(1)$.

(d) Suppose that $f(x) = 1 - \int_2^x f(x)$. Find $f(x)$.

15. Evaluate the following integrals.

$$\int_0^4 \sqrt{1+2x}dx; \qquad \int \sin\left(x^2\right) x dx; \qquad \int_0^{\pi/2} (\cos s)^3 \, ds; \qquad \int \frac{\ln(2x+4)}{x+2} dx$$

$$\int \sin^2(2x) dx; \qquad \int_0^{\pi/2} \sin(x)\cos(x)^2 dx.$$

16. Sketch the region bounded by the curves $x = 2y^2$ and $x + y = 1$. Find its area.