MATH 1500, Homework 1

Due: Monday, 21 September. Questions that say "bonus" are optional; however bonus marks will be given if they are correctly attempted.

1. In class, we used the Newton's method to compute $\sqrt{2}$. It consists of an iteration

$$x_{n+1} = \frac{x_n^2 + 2}{2x_n}. \tag{1}$$

There are other iterations that converge to $\sqrt{2}$. One such example is

$$x_{n+1} = \frac{x_n + 2}{x_n + 1}. \tag{2}$$

   (a) Starting with $x_0 = 2$, apply the iteration (1) several times. How many iterations did you need to get 8 decimal places correctly?

   (b) Starting with $x_0 = 2$, apply the iteration (2) 10 times. How many correct decimal places did you get?

   (c) Based on your observations in parts (a) and (b), estimate how many iterations you will need to get 1000 decimal places of$\sqrt{2}$ using either (1) or (2).

   (d) Solve the equation $x = \frac{x+2}{x+1}$. What does this tell you about (2)?

2. Ever wondered how computers perform division? One way is to use Newton's method. The idea is that the root of $f(x) = a - \frac{1}{x}$ is precisely $1/a$.

   (a) Apply Newton's method to $f(x) = a - \frac{1}{x}$ to come up with an algorithm that allows you to find $1/a$ using only multiplication and addition/substraction. [Recall from high school: $f'(x) = \frac{1}{x^2}$.]

   (b) Using a calculator, illustrate your algorithm to compute $\frac{1}{7}$. Use $x_0 = 0.1$ as your starting point. How many iterations were required to get 8 digits? How many multiplication and addition operations?

   (c) Illustrate graphically and/or algebraically what happens if you use $x_0 = 1$ as your starting point for part (b).

3. [BONUS] The set of all algebraic numbers $\mathcal{A}$ is defined to be a set of all roots of all polynomials with integer coefficients. That is,

$$\mathcal{A} = \{x : a_n x^n + a_{n-1}x^{n-1} + ... + a_0 = 0, \quad n \in \mathbb{N},\ a_0, a_1, ...a_n \in \mathbb{Z}.\}$$

   (a) Show that $\sqrt{2}, \sqrt{3} + \sqrt{2}, \left(\sqrt{3} + \sqrt{2} + 1\right)^{1/3} \in \mathcal{A}$.

   (b) Show that $\mathcal{A}$ is a countable set.

   Note: a real number that is not algebraic is called transcendental. Since $\mathcal{A}$ is countable but $\mathbb{R}$ is not, there are uncountably many transcendental numbers. Despite their abundance, it is difficult to come up with even one simple example of a transcendental number.

4. [BONUS] Show that there uncountably many letters "L", all of the same size, that can be fit into a plane without intersecting each-other. Show that there is only countably many letters "O" for which this can be done. What about the letter "X"? What if you are allowed to resize it?